SOLUTIONS FOR HOMEWORK 4

5.2.10. We have to prove the existence of distinct i and j s.t. both $(x_i + x_j)/2$ and $(y_i + y_j)/2$ are integer, which happens if and only if both $x_i + x_j$ and $y_i + y_j$ are even. However, $a + b$ is even (with $a, b \in \mathbb{Z}$) iff a and b have the same parity (that is, either both are even, or both are odd). Thus, we have to show the existence of $i < j$ with the property that x_i and x_j have the same parity, and the same is true for y_i and y_j. For $1 \leq i \leq j$, let

$$a_i = \begin{cases} 1 & x_i \text{ is odd} \\ 0 & x_i \text{ is even} \end{cases}, b_i = \begin{cases} 1 & y_i \text{ is odd} \\ 0 & y_i \text{ is even} \end{cases}.$$

By the product rule of counting, there are at most four different pairs (a_i, b_i) (there are two ways to pick each of the elements). By the pigeon-hole principle, $(a_i, b_i) = (a_j, b_j)$ for some $i \neq j$. Then, $x_i + x_j$ and $y_i + y_j$ are even, hence the required condition on the midpoints.

5.2.14. **(a)** Break all the integers from 1 to 10 into pairs of the form $(k, 11 - k)$, with $1 \leq k \leq 5$: that is, we get the pairs $(1, 10), (2, 9), \ldots, (5, 6)$. Each integer between 1 and 10 belongs to exactly one of these five pairs. Suppose 7 numbers have been selected. Then, in at least two pairs, both members have been picked. Indeed, otherwise we have at most one pair with both members selected, and four pairs with at most one members selected. Hence, no more than $2 \cdot 1 + 1 \cdot 4 = 6$ numbers have been picked, a contradiction.

(b) The conclusion no longer holds. Indeed, suppose we pick $1, 2, \ldots, 6$. Then there is only one pair adding up to 11: $5 + 6 = 11$.

5.2.32. Denote by n_k the number of connections of k-th computer ($1 \leq k \leq 6$). Then $1 \leq n_k \leq 5$. As the set $\{1, 2, 3, 4, 5\}$ has five elements, the pigeon-hole principle implies that $n_i = n_j$ for some $i \neq j$ (here, the n_k's are pigeons, while $1, \ldots, 5$ enumerate the holes).

5.3.28. An answer key is a sequence of 40 letters: 17 Ts and 23 Fs. Different answer keys corresponds to different ways to place the Fs into the 40 positions. Thus, the number of different answer keys equals the number of ways to select 17 objects out of 40, when the order of selection is not important. Thus, the answer is $\binom{40}{17}$.

5.3.30. **(b)** Let n be the total number of ways to select a committee of five members (out of 9 men and 7 women). Let n_0, n_m, and n_w be the number of ways to select a committee such that:

We are interested in $n_0 = n - n_m - n_w$. However, $n = \binom{16}{5}$ (the number of ways to

n_w	no women are present
n_m	no men are present
n_0	both men and women are present

pick 5 objects out of 16), $n_m = \binom{7}{5}$, and $n_w = \binom{9}{5}$. Then

$$n_0 = \binom{16}{5} - \binom{7}{5} - \binom{9}{5}.$$

5.3.38. Note that the following holds:

Number of ways to select 3 countries out of 45	$\binom{45}{3}$
Number of ways to select 4 countries out of 57	$\binom{57}{4}$
Number of ways to select 5 countries out of 69	$\binom{69}{5}$

By the product rule, the total number of ways to select 12 countries equals $\binom{45}{3}\binom{57}{4}\binom{69}{5}$.

Problem A. *This is a bonus problem. Very little partial credit will be given.* Suppose we are given n integers. Prove that the sum of several ("several" may be $1, 2, \ldots, n$) of them is divisible by n.

We refer to our integers as $a_1, \ldots, a_n$. Denote by r_k ($1 \leq k \leq n$) the remainder obtained when $a_1 + \ldots + a_k$ is divided by n. In other words,

$$a_1 + \ldots + a_k = nb_k + r_k,$$

where $b_k \in \mathbb{Z}$, and $0 \leq r_k \leq n-1$. If $r_k = 0$ for some k, then n divides $a_1, \ldots, a_k$, and we are done. If there is no k satisfying $r_k = 0$, then all the n remainders r_k belong to the set $\{1, 2, \ldots, n-1\}$ with $n-1$ elements. Therefore, $r_i = r_j$ for some $i < j$. Then

$$a_{i+1} + \ldots + a_j = (a_1 + \ldots + a_j) - (a_1 + \ldots + a_i) = (nb_j + r_j) - (nb_i + r_i) = n(b_j - b_i)$$

is divisible by n, which is what we need.

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