SOLUTIONS FOR HOMEWORK 5

5.4.10 We can write

$$\left(x+\frac{1}{x}\right)^{100} = \sum_{k=0}^{100} \binom{100}{k} x^k \left(\frac{1}{x}\right)^{100-k} = \sum_{k=0}^{100} \binom{100}{k} x^{100-2k}.$$

The number $100-2k$ is always even, and lies between -100 and 100. Therefore, the coefficient of x^s of $(x+1/x)^{100}$ equals 0 if s is odd, or $|s| > 100$. If $s \in [-100, 100]$ is even, write $s = 100-2k$, with $k = 50 - s/2$. Then the coefficient of x^s will be equal to $\binom{100}{50-s/2}$.

5.4.14 We know that $\binom{n}{k} = \binom{n}{n-k}$. Therefore, it suffices to show that $\binom{n}{k-1} < \binom{n}{k}$ for $k \leq n/2$. But then, $n-k+1 > k$. hence

$$\binom{n}{k-1} = \frac{n \cdot \ldots \cdot (n-k+2)}{(k-1)!}$$
$$< \frac{n \cdot \ldots \cdot (n-k+2)}{(k-1)!} \cdot \frac{n-k+1}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} = \binom{n}{k}.$$

5.4.16. **(a)** Note that, for $n > 1$,

$$\binom{n}{0} + \binom{n}{n} = 2 \leq n = \binom{n}{1}.$$

By Exercise 14, $\binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{k}$ for any k. Furthermore,

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \left(\binom{n}{0} + \binom{n}{n}\right) + \sum_{k=1}^{n-1} \binom{n}{k} \leq n \binom{n}{\lfloor n/2 \rfloor},$$

which implies $\binom{n}{\lfloor n/2 \rfloor} \geq 2^n/n$.

(b) Setting $n = 2m$, obtain $\binom{2m}{m} \geq 2^{2m}/(2m)$.

5.4.22. **(a)** Compute the number of ways to select a team of r members out of n employees, and, within this team, to pick k managers.

On the one hand, we get $\binom{n}{r}\binom{r}{k}$. The first term in the product corresponds to selecting r members, and the second term, to selecting the managers out of these k team members.

On the other hand, we can select the managers first (there are $\binom{n}{k}$ ways to do so), and then pick then $r-k$ non-managerial team members ($\binom{n-k}{r-k}$ ways). This gives us the total number of possible selections $\binom{n}{k}\binom{n-k}{r-k}$.

Therefore, $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

(b)

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!} \cdot \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{(n-r)!(r-k)!} = \binom{n}{k}\binom{n-k}{r-k}.$$

5.4.30 Let's compute the number of ways to form a committee of n professors ($k \in \{1, 2, \dots, n\}$ from Math, $n-k$ from CS), and the committee's Chair (from Math), if we have n Math professors, and n CS professors.

On the one hand, the number of ways to pick a committee of k professors from Math and $n-k$ professors from CS equals $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. There k ways to pick a Chair for such a committee. Thus, when k is given, the total number of selections is $k\binom{n}{k}^2$. As k can be between 1 and n, the total number of possible selections equals $\sum_{k=1}^{n} k\binom{n}{k}^2$.

On the other hand, we have n ways to pick the Chair, and $\binom{2n-1}{n-1}$ ways to select the remaining $n-1$ members. Therefore,

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

5.5.10. There are 6 kinds of croissants. Throughout, we shall denote the number of croissants of k-th kind in our selection by i_k.

(a) The number of 12-combinations of 6 objects with repetitions equals

$$\binom{12+6-1}{12} = \binom{17}{12} = \binom{17}{5}.$$

This is the same as the number of ways to write $12 = i_1 + \ldots + i_6$, where $i_1, \ldots, i_6$ are non-negative integers.

(c) Here, we are looking for number of solutions of the equation $24 = i_1 + \ldots + i_6$, where $i_1, \ldots, i_6$ are integers, no less than 2. For $1 \leq s \leq 6$, let $j_s = i_s - 2$. Then we have to compute the number of ways to represent $12 = j_1 + \ldots + j_6$, where $j_1, \ldots, j_6$ are non-negative integers. As we saw in (a), the number of such solutions equals $\binom{17}{12} = \binom{17}{5}$.

(d) The total number of ways to pick 24 croissants equals $n_1 = \binom{24+6-1}{24} = \binom{29}{24} = \binom{29}{5}$ (the number of 24-combinations of 6 objects, with repetitions). The number of ways to select 24 croissants in such a way that at least 3 of them are broccoli equals the number of solutions of the equation $24 = i_1 + \ldots + i_6$, where $i_1, \ldots, i_6$ are non-negative integers, and $i_6 \geq 3$. This, in turn, equals the number of non-negative integer solutions of $21 = j_1 + \ldots + j_6$ ($i_s = j_s$ for $1 \leq s \leq 5$, and $j_6 = i_6 - 3$). The latter equals $n_2 = \binom{21+6-1}{21} = \binom{26}{21} = \binom{26}{5}$.

The number of selections which include no more than 2 broccoli croissants equals $n = n_1 - n_2 = \binom{29}{5} - \binom{26}{5}$.

Alternative solution. Let m_r be the number of selections with r broccoli croissants present (in other words, $i_6 = r$). This equals the number of non-negative integer solutions of the equation $24 = i_1 + \ldots + i_5 + r$, or equivalently, $24 - r = i_1 + \ldots + i_5$.

The number of such solutions equals the number of $(24-r)$-combinations of 5 objects with repetitions, or $\binom{24-r+5-1}{24-r} = \binom{28-r}{24-r} = \binom{28-r}{4}$. The number of selections which include no more than 2 broccoli croissants equals

$$n = m_0 + m_1 + m_2 = \binom{28}{4} + \binom{27}{4} + \binom{26}{4}.$$

5.5.20. Let $x_4 = 11 - x_1 - x_2 - x_3$. Then we are interested in the number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 11$. This equals the number of ways to distribute 11 indistinguishable balls into 4 boxes ($x_1, \ldots, x_4$ refer to the quantity of balls in a given box). In other words, we have to compute the number of 11-combinations of 4 objects with repetitions. The latter is $\binom{11+4-1}{11} = \binom{14}{11} = \binom{14}{3}$.

5.5.30. The word *MISSISSIPPI* contains 1 M, 4 I's, 4 S's, and 2 P's (11 letters total). This corresponds to the problem of placing 11 distinguishable objects (integers from 1 to 11) into four boxes, labeled M (must contain 1 object), I (contains 4 objects), S (contains 4 objects), and P (contains 2 objects). The placing of objects into a box (say, S) corresponds the letter S positions (4 out of 11). The number of such arrangements equals $11!/(1!4!4!2!)$.

5.5.38. **(a)** We place $n = 40$ distinguishable items into 4 distinguishable boxes, so that the i-th box ($i = 1, 2, 3, 4$) contains $n_i = 10$ items. There are

$$N = \frac{n!}{n_1!n_2!n_3!n_4!} = \frac{40!}{(10!)^4}$$

ways to do it.

(b) Denote by M the number of ways to place 40 distinguishable objects into 4 indistinguishable boxes, 10 objects per box. Once this has been accomplished, there are 4! ways to enumerate the boxes. Thus, $N = 4!M$. Indeed, any placing of 9 objects in 4 boxes, enumerated by $1, 2, 3, 4$ (10 objects per box) can be accomplished by first placing the objects into unlabeled boxes, and then numbering the boxes. Taking the expression for N from part (a), we see that

$$M = \frac{40!}{4!(10!)^4}.$$

5.5.46. We need to compute the number of ways to arrange 5 $|$s and 7 $*$s in a row, in such a way that no to $|$ are adjacent. Denote the number of such arrangements by n. Let n_1 (n_2) be the number of such arrangements where the leftmost elements is a $|$, resp. a $*$. Then $n_1 + n_2 = n$.

Compute n_2 first. In each of these arrangements, a $|$ is preceded by a $*$. Thus, n_2 is the number of was to arrange 5 As and 2 $*$s in a row, where A stands for $*|$. Therefore, $n_2 = 7!/(5!2!)$.

Next compute n_1. Throwing away the first $|$, we need to count the number of ways to arrange 7 $*$s and 4 $|$s in a row, in such a way that every $|$ is preceded by a

$*$. Substituting A for $*|$ (as in the previous paragraph), we see that n_1 equals the number of ways to arrange 4 As and 3 $*$ in a row, which equals $7!/(4!3!)$.

Therefore,

$$n = n_1 + n_2 = 7!\Big(\frac{1}{5!2!} + \frac{1}{4!3!}\Big).$$

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