SOLUTIONS FOR HOMEWORK 6

6.1.12. A deck contains 52 cards, four of them aces. The total number of poker hands (that is, the number of collections of 5 cards) equals $\binom{52}{5}$. This is the cardinality of our sample space S.

The number of hands containing exactly one ace equals the number of ways to pick one ace (out of four), and four other cards. There are 4 ways to select the ace, and $\binom{48}{4}$ ways to pick the other cards. By the product rule, the number of hands with exactly one ace equals $4\binom{48}{4}$. This is the cardinality of the even E we are interested in.

Thus, $p(E) = |E|/|S| = 4\binom{48}{4}/\binom{52}{5}$.

6.1.14. Our sample space S is the set of all selections of 5 cards (our of 52). As in the previous exercise, $|S| = \binom{52}{5}$. The even E occurs if 5 different kinds of cards (out of 13) are present. There are $\binom{13}{5}$ ways to select the kinds of cards. For each of the 5 selected kinds, there are four ways to pick a suit. Thus, $|E| = 4^5\binom{13}{5}$, and $p(E) = |E|/|S| = 4^5\binom{13}{5}/\binom{52}{5}$.

6.2.12. Clearly, $p(X) \leq p(Y)$ if $X \subseteq Y$. Therefore, $p(E \cup F) \geq p(E) = 0.8$.

On the other hand,

$$1 \geq p(E \cup F) = p(E) + p(F) - p(E \cap F) = 0.8 + 0.6 - p(E \cap F),$$

hence $p(E \cap F) \geq 0.8 + 0.6 - 1 = 0.4$.

6.2.16. We prove first: if E and F are independent, then E and $\overline{F}$ are independent. Indeed, it is easy to see that the sets $E \cap \overline{F}$ and $E \cap F$ are disjoint, and their union equals E. Therefore, $p(E) = p(E \cap \overline{F}) + p(E \cap F)$. The events E and F are independent iff then $p(E \cap F) = p(E)p(F)$. In this case,

$$p(E \cap \overline{F}) = p(E) - p(E \cap F) = p(E) - p(E)p(F) = p(E)\big(1 - p(F)\big) = p(E)p(\overline{F}),$$

which means E and $\overline{F}$ are independent. Similarly, we show that $\overline{E}$ and $\overline{F}$ are independent.

6.2.18. We enumerate the days of the week: $1, 2, \ldots, 7$.

(a) The sample space S is the set of ordered pairs (a, b), where a and b are integers between 1 and 7. Then $|S| = 7^2$. The even $E = \{(a, a) : 1 \leq a \leq 7,\ a \in \mathbb{Z}\}$, hence $|E| = 7$, and $p(E) = 1/7$.

(b) We only consider the case of $2 \leq n \leq 7$. Indeed, if $n \geq 7$, then the Pigeon-Hole Principle implies that two people (pigeons) were born on the same day of the week (have to share a hole). The sample space S consists of all ordered n-tuples of integers between 1 and 7, hence $|S| = 7^n$. The event E we are interested in consists of the

n-tuples containing at least two identical entries. Then $\overline{E}$ consists of the n-tuples where all the entries are distinct. Then $|\overline{E}| = P(7,n) = 7 \cdot 6 \cdot \ldots \cdot (8-n)$, and

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{P(7,n)}{7^n} = 1 - \frac{6 \cdot \ldots \cdot (8-n)}{7^{n-1}}.$$

(c) We need to find n for which $p(E) \geq 1/2$. When $n = 4$, $p(E) = 0.65$. When $n = 3$, $p(E) = 0.39$. Thus, we need to pick 4 persons. 3 is not enough.

6.2.24. Let E be the event that 4 heads appear on 5 throws of a fair coin, and let F be the event that tails appears on the first toss. Our sample space consists of all sequences of 5 Hs and Ts. All these sequences are equally likely. We have $|S| = 2^5$ (each entry in the 5-tuple can be selected in two different ways). $|F| = 2^4$ (the first entry in the 5-tuple is T, there are two ways to pick each of the remaining four). Furthermore, $|E \cap F| = 1$: this even consists of only one 5-tuple, namely, $(THHHH)$. So, $p(E \cap F) = |E|/|S| = 1/32$, $p(F) = |F|/|S| = 16/32$, hence $p(E|F) = p(E \cap F)/p(F) = 1/16$.

6.2.26. The sample space S consists of the strings of 0s and 1s of length 3. $|S| = 2^3$. Let E be the event that the number of 1s is odd, and F be the event that 1 is the starting digit. Then $|F| = 2^2$ (the first digit is fixed, each of the other 2 can be selected in 2 ways). For $0 \leq k \leq 3$, let A_k be the event that exactly k 1s appear. Then $|A_k| = \binom{3}{k}$ (pick k positions for 1s, out of 3). It is easy to see that $E = A_1 \cup A_3$, hence $|E| = |A_1| + |A_3| = 4$. Moreover, $E \cap F = \{100, 111\}$, hence $|E \cap F| = 2$. Therefore, $p(E \cap F) = 2/8 = 1/4$, $p(E) = 4/8 = 1/2$, and $p(F) = 4/8 = 1/2$. Therefore, $p(E \cap F) = p(E)p(F)$. In other words, E and F are independent.

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