SOLUTIONS FOR HOMEWORK 8

6.4.28. *This is a bonus problem – very little partial credit is given.*
Recall the following fact: for $a_1, \dots, a_n \in \mathbb{R}$,

$$\big(a_1 + \dots + a_n\big)^2 = \sum_{k=1}^{n} a_k^2 + 2 \sum_{1 \le k < \ell \le n} a_k a_\ell.$$

Let $X = \sum_{k=1}^n X_k$. Then $X^2 = \sum_{k=1}^n X_k^2 + 2\sum_{1\le k<\ell\le n} X_k X_\ell$. By the linearity of the expectation, and by the pairwise independence of the random variables $X_1, \dots, X_n$,

$$\mathbb{E}(X^2) = \sum_{k=1}^{n} \mathbb{E}(X_k^2) + 2 \sum_{1 \le k < \ell \le n} \mathbb{E}(X_k)\mathbb{E}(X_\ell).$$

Moreover, $\mathbb{E}(X) = \sum_{k=1}^n \mathbb{E}(X_k)$, hence

$$(\mathbb{E}(X))^2 = \sum_{k=1}^{n} \mathbb{E}(X_k)^2 + 2 \sum_{1 \le k < \ell \le n} \mathbb{E}(X_k)\mathbb{E}(X_\ell).$$

Thus,

$$\mathrm{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sum_{k=1}^{n} \big(\mathbb{E}(X_k^2) - \mathbb{E}(X_k)^2\big) = \sum_{k=1}^{n} \mathrm{V}(X_k).$$

6.4.38. By the linearity of expectation,

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}\big((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\big) \\ &= \mathbb{E}\big(XY - \mathbb{E}(Y)X - E(X)Y + \mathbb{E}(X)\mathbb{E}(Y)\big) \\ &= \mathbb{E}(X) - \mathbb{E}(Y)\mathbb{E}(X) - \mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(X) - \mathbb{E}(X)\mathbb{E}(Y).\end{aligned}$$

6.4.39. By Theorem 6, $\mathrm{V}(X+Y) = \mathbb{E}\big((X+Y)^2\big) - \big(\mathbb{E}(X+Y)\big)^2$. By the linearity of conditional expectation,

$$\mathbb{E}\big((X+Y)^2\big) = \mathbb{E}(X^2 + 2XY + Y^2) = \mathbb{E}(X^2) + \mathbb{E}(Y^2) + 2\mathbb{E}(XY),$$

and

$$\big(\mathbb{E}(X+Y)\big)^2 = \big(\mathbb{E}(X) + \mathbb{E}(Y)\big)^2 = \big(\mathbb{E}(X)\big)^2 + \big(\mathbb{E}(Y)\big)^2 + 2\mathbb{E}(X)\mathbb{E}(Y).$$

Therefore,

$$\begin{aligned}\mathrm{V}(X+Y) &= \big(\mathbb{E}(X^2) - \mathbb{E}(X)^2\big) + \big(\mathbb{E}(Y^2) - \mathbb{E}(Y)^2\big) + 2\big(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)\big) \\ &= \mathrm{V}(X) + \mathrm{V}(Y) + 2\mathrm{Cov}(X,Y).\end{aligned}$$

6.4.41. We can assume the bins are enumerated: $1, 2, \dots, n$. Then the sample space S consists of all m-tuples, whose entries are integers between 1 and n: the i-th entry of the m-tuple equals k if the i-th ball fall into the k-th bin. Then $|S| = n^m$. The event E (the first bin is empty) is set set of the m-tuples made up of integers $2, \dots, n$. Then $|E| = (n-1)^m$, hence $p(E) = |E|/|S| = (1-1/n)^m$.

6.4.42. Denote by X the number of balls falling into the first bin. For $1 \le i \le m$, let $X_i = 1$ if the i-th ball falls into the first bin, and $X_i = 0$ otherwise. As in Exercise 6.4.41, our sample space S consists of all m-tuples, whose entries are integers between 1 and n. Then $|S| = n^m$. Moreover, the event $\{X_i = 1\}$ consists of the m-tuples where the i-th entry equals 1. Then $|\{X_i = 1\}| = n^{m-1}$, hence $p(X_i = 1) = 1/n$. Furthermore, $p(X_i = 0) = 1 - p(X_i = 1) = 1 - 1/n$, hence $\mathbb{E}(X_i) = 1/n$.

Note that $X = \sum_{i=1}^m X_i$. Thus, $\mathbb{E}(X) = \sum_{i=1}^m \mathbb{E}(X_i) = m/n$.

6.4.43. Denote by Y the total number of bins (out of n) that remain empty. For $1 \le j \le n$, let $Y_j = 1$ if the j-th bin is empty, and $Y_j = 0$ otherwise. Then $Y = \sum_{j=1}^n Y_j$. By Exercise 6.4.41, $p(Y_j = 1) = (n-1)^m/n^m$, hence $\mathbb{E}(Y_j) = (n-1)^m/n^m$. Then $\mathbb{E}(Y) = \sum_{j=1}^n \mathbb{E}(Y_j) = (n-1)^m/n^{m-1}$.

Remark. Suppose $n = m$ (the number of bins equals the number of balls). For large values of n, $(1-1/n)^n \simeq 1/e$, where $e = 2.718....$ The expected number of empty bins is approximately equal $n/e \simeq 0.37n$.

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