SOLUTIONS FOR HOMEWORK 9

7.1.4. We have to show that each of the following sequences solves the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$.

(a) $a_n = 0$: $0 = -3 \cdot 0 + 4 \cdot 0$.

(b) $a_n = 1$: $1 = -3 \cdot 1 + 4 \cdot 1$.

(c) $a_n = (-4)^n$: we have to show that $(-4)^n = -3(-4)^{n-1} + 4(-4)^{n-2}$ for any $n \geq 2$. But the left hand side equals

$$-3(-4)^{n-1} + 4(-4)^{n-2} = (-4)^{n-2}\big((-3)(-4) + 4\big) = 16(-4)^{n-2} = (-4)^2(-4)^n,$$

which is what we need to obtain on the right.

(d) $a_n = 2(-4)^n + 3$: we can check the recurrence relation directly, as in (c). Alternatively, we can write $a_n = 2b_n + 3c_n$, where $b_n = (-4)^n$ and $c_n = 1$ for each $n \geq 1$. We have already verified (in (c) and (b), respectively) that the sequences (b_n) and (c_n) solve our recurrence relation. We showed in class that the sequence $a_n = \beta b_n + \gamma c_n$ solves our (homogeneous!) recurrence relation for any $\beta, \gamma \in \mathbb{R}$.

7.1.8. **(d)** $a_0 = -1$, $a_n = 2a_{n-1} - 3$. Iterating, we see that

$$a_n = 2a_{n-1} - 3 = 2(2a_{n-2} - 3) - 3 = 2^2 a_{n-2} - 3(2 + 1).$$

Proceeding further in the same manner, we obtain:

$$\begin{aligned} a_n &= 2^3 a_{n-3} - 3(2^2 + 2 + 1) = \dots \\ &= 2^{n-1} a_1 - 3\big(2^{n-1} + 2^{n-2} + \dots + 2 + 1\big) = 2^n(-1) - 3\big(2^n - 1\big) = 3 - 2^{n+2} \end{aligned}$$

(here, we use the formula for the sum of a geometric sequence, as well as $2^n + 3 \cdot 2^n = 4 \cdot 2^n = 2^2 2^n = 2^{n+2}$).

We can (although we do not have to) use induction to check that, indeed, $a_n = 3 - 2^{n+2}$ for any non-negative integer n. The basic step is handled by verifying the above equation for $n = 0$. For the inductive step, show that $a_{n+1} = 3 - 2^{n+1}$ if $a_n = 3 - 2^{n+2}$. By the induction hypothesis,

$$a_{n+1} = 2a_n - 3 = 2\big(3 - 2^{n+2}\big) - 3 = 2 \cdot 3 - 2^{n+3} - 3 = 3 - 2^{n+1},$$

which is what we need.

7.1.22. Denote by x_n the number of strictly increasing sequences of integers whose first and last terms equal 1 and n, respectively, that is, the number of sequences $1 \leq a_1 < a_2 < \dots < a_{k-1} < a_k = n$.

(a) For $2 \leq m \leq n-1$, among the increasing sequences as above there exist exactly x_m sequences with $a_{k-1} = m$. Moreover, there exists one sequence where $a_{k-1} = a_1 = 1$. Therefore, $x_n = 1 + x_2 + x_3 + \dots + x_{n-1}$.

(b) $x_2 = 1$.

(c) $x_n = 2^{n-2}$. This can be proved by strong induction. Indeed, for $n = 2$ this formula is true, by (b). This is the basic step. To handle the inductive step, suppose $x_k = 2^{k-2}$ for $2 \leq k \leq n$, and prove that $x_{n+1} = 2^{n-1}$. By (a) and the induction hypothesis,

$$x_{n+1} = 1 + \sum_{k=2}^{n} x_k = 1 + \sum_{k=2}^{n} 2^{k-2} = 1 + \big(2^{n-1} - 1\big) = 2^{n-1}.$$

This completes the inductive step.

Alternative solution. There is a one-to-one correspondence between increasing sequences $1 = a_1 < a_2 < \ldots < a_{k-1} < a_k = n$, and subsets of $\{2, \ldots, n-2\}$. Indeed, a sequence as above determines a set $\{a_2, \ldots, a_{k-1}\}$ (this set is empty if $k = 2$). On the other hand, $S \subset \{2, \ldots, n-2\}$ determines an increasing sequence of the type we need: just enumerate S in the increasing order, and throw in $a_1 = 1$ and $a_k = n$. Now, recall that there are 2^{n-2} subsets of $\{2, \ldots, n-2\}$.

7.1.44. The Fibonacci sequence is determined by the recurrence relation $f_n = fn - 1 + f_{n-2}$ $(n \geq 2)$, and by the initial condition $f_0 = 0$, $f_1 = 1$. Then, for $n \geq 3$,

$$f_n = f_{n-1} + f_{n-2} = \big(f_{n-2} + f_{n-3}\big) + f_{n-2} = 2f_{n-2} + f_{n-3}.$$

Furthermore, for $n \geq 5$,

$$f_n = 2f_{n-2} + f_{n-3} = 2\big(2f_{n-4} + f_{n-5}\big) + \big(f_{n-4} + f_{n-5}\big) = 5f_{n-4} + 3f_{n-5}.$$

Thus, $f_n = 5f_{n-4} + 3f_{n-5}$ for any $n \geq 5$. Using this recurrence relation with $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and $f_4 = 3$ (obtained from the usual formula for the Fibonacci numbers), we recover the Fibonacci sequence.

Now, we can use mathematical induction to prove that f_{5m} is divisible by 5 for every non-negative integer m. The basic step is the case $m = 0$, $f_0 = 0$. To complete the inductive step, we have to prove that $f_{5(m+1)}$ is divisible by 5, provided f_{5m} is. But we have shown that $f_{5(m+1)} = 5f_{5m+1} + 3f_{5m}$. If f_{5m} is divisible by 5, then so is the left hand side.

7.2.4. (a) We have to solve the linear homogeneous recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ $(n \geq 2)$ with the initial conditions $a_0 = 3$, $a_1 = 6$. Start by solving the quadratic equation $r^2 - r - 6 = 0$. Its roots are $r_1 = 3$ and $r_2 = -2$. Then

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \tag{1}$$

for any $n \geq 0$. The scalars α_1 and α_2 are chosen in such a way that (1) holds for $n = 0, 1$. Thus,

$$\left\{ \begin{array}{ccccc} 3 & = & a_0 & = & \alpha_1 + \alpha_2 \\ 6 & = & a_1 & = & 3\alpha_1 - 2\alpha_2 \end{array} \right. .$$

Solving this system of linear equations, we obtain: $\alpha_1 = 12/5$, $\alpha_2 = 3/5$. Going back to (1), we conclude that $a_n = (12 \cdot 3^n + 3 \cdot (-2)^n)/5$.

7.2.8. (a) $L_n = L_{n-1}/2 + L_{n-2}/2$ for $n \geq 2$.

(b) The linear homogeneous recurrence relation above gives rise to the equation $r^2 - r/2 - 1/2 = 0$ (that is, $k = 2$, and $c_1 = c_2 = 1/2$). Solving this equation, we see that its roots are $r_1 = 1$ and $r_2 = -1/2$. Then any solution of the recurrence relation is of the form $L_n = \alpha_1 + \alpha_2/2^n$.

From the initial conditions, we have:

$$\begin{cases} 10^5 & = & L_1 & = & \alpha_1 - \alpha_2/2 \\ 3 \cdot 10^5 & = & L_2 & = & \alpha_1 + \alpha_2/4 \end{cases}.$$

Solving this system of linear equations, we see that $\alpha_2 = -8 \cdot 10^5/3$, and $\alpha_1 = 7 \cdot 10^5/3$. Thus, $L_n = (7 - 8/(-2)^n)10^5/3$ fr any n.

7.2.14. We have to find the solution to the linear homogeneous recurrence relation $a_n = 5a_{n-2} - 4a_{n-4}$, satisfying the initial condition $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, $a_3 = 8$. The recurrence relation gives rise to the characteristic equation $r^4 - 5r^2 + 4 = 0$. Letting $s = r^2$, we have: $s^2 - 5s + 4 = 0$. Solving this quadratic equation, we conclude that s equals to 1 or 4. Thus, the characteristic equation $r^4 - 5r^2 + 4 = 0$ has roots (in the increasing order $r_1 = -2$, $r_2 = -1$, $r_3 = 1$, and $r_4 = 2$. Therefore, any solution to the recurrence relation is of the form $a_n = \alpha_1(-2)^n + \alpha_2(-1)^n + \alpha_3 + \alpha_4 2^n$. The coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are to be found from the initial conditions:

$$\begin{cases} 3 & = & a_0 & = & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ 2 & = & a_1 & = & -2\alpha_1 - \alpha_2 + \alpha_3 + 2\alpha_4 \\ 6 & = & a_2 & = & 4\alpha_1 + \alpha_2 + \alpha_3 + 4\alpha_4 \\ 8 & = & a_3 & = & 8\alpha_1 - \alpha_2 + \alpha_3 + 8\alpha_4 \end{cases}. \tag{2}$$

Considering the equations for a_0 and a_2 in (2), we have:

$$\begin{cases} 3 & = & (\alpha_1 + \alpha_4) + (\alpha_2 + \alpha_3) \\ 6 & = & 4(\alpha_1 + \alpha_4) + (\alpha_2 + \alpha_3) \end{cases},$$

hence $\alpha_1 + \alpha_4 = 1$, and $\alpha_2 + \alpha_3 = 2$. Similarly, considering the equations for a_1 and a_3 in (2), we obtain:

$$\begin{cases} 2 & = & 2(-\alpha_1 + \alpha_4) + (-\alpha_2 + \alpha_3) \\ 8 & = & 8(-\alpha_1 + \alpha_4) + (-\alpha_2 + \alpha_3) \end{cases},$$

which implies $-\alpha_1 + \alpha_4 = 1$, and $-\alpha_2 + \alpha_3 = 0$.

We can find α_1 and α_4 from the system of two linear equations

$$\begin{cases} -\alpha_1 + \alpha_4 & = & 1 \\ \alpha_1 + \alpha_4 & = & 1 \end{cases}.$$

Solving it, we get: $\alpha_1 = 0$, $\alpha_4 = 1$. Similarly, α_2 and α_3 are determined by the system

$$\begin{cases} -\alpha_2 + \alpha_3 & = & 0 \\ \alpha_2 + \alpha_2 & = & 2 \end{cases}.$$

Thus, $\alpha_2 = \alpha_3 = 1$. Therefore, $\mathbf{a_n = (-1)^n + 1 + 2^n}$.

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