MATH 213 MIDTERM 1: SOLUTIONS FOR PRACTICE PROBLEMS

The test will be given on **Wednesday, October 8**. It will be based on Homeworks 1-5 (Sections 2.1-3, 4.1-2, 5.1-5).

In preparing for the test, you can practice solving the problems from the list below. In addition, take a look at the homework problems (at least one problems on the midterm will come directly from the homework), and at the examples given in the textbook.

1. Prove that $(A \cap B) \cup (A - B) = A$.

It follows immediately from the definition that $A - B = A \cap \overline{B}$. To show that $A = (A \cap B) \cup (A - B)$, use the distributive law for the union of sets:

$$(A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) = A \cap U = A.$$

2. Prove that, for any non-negative integer n, 3 divides $2 \cdot 4^n + 1$.

For $n \geq 0$, let $P(n)$ be the statement: 3 divides $2 \cdot 4^n + 1$. We use induction to prove that $P(n)$ is true for every n.

The basic step is easy: $2 \cdot 4^0 + 1 = 3$ is clearly a multiple of 3.

On the inductive step, we have to prove that, if $P(n)$ is true, then $P(n+1)$ is true. That is, we have to prove that if 3 divides $2 \cdot 4^n + 1$, then 3 divides $2 \cdot 4^{n+1} + 1$. Note that $4^{n+1} = 4 \cdot 4^n = 3 \cdot 4^n + 4^n$, hence $2 \cdot 4^{n+1} + 1 = 6 \cdot 4^n + (2 \cdot 4^{n+1} + 1)$. By the induction hypothesis, the right hand side is divisible by 3.

3. A student has 10 apples, 12 pears, and 16 oranges. In how many ways can she select 9 fruits, so that at least 2 pears, and at least 3 oranges, are chosen?

Denote by i_1, i_2, and i_3 the number of apples, pears, and oranges selected. We have to compute the number of integer solutions of the equation $i_1 + i_2 + i_3 = 9$, satisfying $0 \leq i_1 \leq 10$, $2 \leq i_2 \leq 12$, and $3 \leq i_3 \leq 16$. Clearly, the upper bounds on i_1, i_2, and i_3 will be satisfied automatically. Let $j_1 = i_1$, $j_2 = i_2 - 2$, and $j_3 = i_3 - 3$. We are reduced to computing the number of non-negative integer solutions of $j_1 + j_2 + j_3 = 9 - (2 + 3) = 4$. This is equal to the number of 4-combinations of 3 objects with repetitions, that is, $\binom{4+3-1}{4} = \binom{6}{4} = \binom{6}{2} = 15$.

4. How many sequences of five integers between 1 and 9 satisfy the following conditions:

(a) All the numbers are different.

There number of 5-permutations of 9 objects: $P(9, 5) = 9!/4!$.

(b) The sequence contains the number 3 exactly twice.

There are $\binom{5}{2}$ ways to select the positions for the two 3s. Once thees positions have been selected, there are 8^3 ways to fill the remaining 3 spots with numbers different from 3. By the product rule, the total number of outcomes equals $8^3\binom{5}{2}$.

(c) All the five numbers are different, and arranged in the increasing order.

We have to compute the number of ways to select 5 distinct numbers out of $\{1, \ldots, 9\}$. Once these numbers have been selected, there is only one way to arrange them in the increasing order. But the number of all possible selections of 5 numbers equals $\binom{9}{5}$.

(d) The numbers are arranged in the non-increasing order (they need not be distinct).

$\binom{14}{5}$. Indeed, our task is to select 5 objects (numbers) of 9 kinds, with repetitions (the order of selection is not important). There are $\binom{9+5-1}{5} = \binom{14}{5}$ ways to do so. Once he five numbers are picked, there is only one way to arrange them in the non-increasing order.

5. Prove that, for any positive integer n,

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{j=n+1}^{2n} \frac{1}{j}.$$

We prove this statement by induction on n. For $n \in \mathbb{N}$, let $P(n)$ be the following statement:

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{j=n+1}^{2n} \frac{1}{j}.$$

The basic step consists of verifying $P(1)$. This is easy: $1 - 1/2 = 1/2$.

For the inductive step, we have to show that, if

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{j=n+1}^{2n} \frac{1}{j},$$

then

$$\sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} = \sum_{j=n+2}^{2n+2} \frac{1}{j}.$$

By the induction hypothesis,

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \frac{1}{n+1} + \sum_{j=n+2}^{2n} \frac{1}{j},$$

hence

$$\begin{aligned}\sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} &= \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} + \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= \sum_{j=n+2}^{2n} \frac{1}{j} + \frac{1}{2n+1} + \frac{1}{n+1} - \frac{1}{2n+2} = \sum_{j=n+2}^{2n} \frac{1}{j} + \frac{1}{2n+1} + \frac{1}{2n+2},\end{aligned}$$

as desired.

6. The sequence (x_n) is defined by the following rule: $x_1 = x_2 = 1$, and $x_n = x_{n-1} + 2x_{n-2}$ for $n \geq 3$. Prove that, for any $n \in \mathbb{Z}_+$, $x_n \geq 2^{n-2}$.

Let $P(n)$ be the statement that $x_n \geq 2^{n-2}$. We have to show that $P(n)$ is true for any n. The proof proceeds by strong induction. The basic step consists of verifying $P(n)$ for $n = 1, 2$ (which is easy). For the inductive step, we establish the following: if $P(k)$ is true for $1 \leq k \leq n$ ($n \geq 2$), then $P(n+1)$ is also true. In other words, we have to show that $x_{n+1} \geq 2^{n-1}$, assuming that $x_k \geq 2^{k-1}$ for any $k \leq n$. By the induction hypothesis,

$$x_{n+1} = x_n + 2x_{n-1} \geq 2^{n-2} + 2 \cdot 2^{n-3} = 2^{n-2} + 2^{n-2} = 2^{n-1}.$$

7. Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$, defined by $f(a, b) = |a - b^2|$.

(a) Is f one-to-one? **NO**. $f(0, 0) = f(1, 1)$.

(b) Is f onto? **YES**. For any $c \in \mathbb{N}$, $f(c, 0) = c$.

8. How many ways are there to re-arrange the letters of the word RESIST is such a way that the R and the T are not adjacent?

Denote by n the number of ways to rearrange the letters so that the R and the T are not adjacent, by n_0 the total number of rearrangements, and by n_1 the number of rearrangements in which the R and the T are adjacent. Then $n = n_0 - n_1$.

The word RESIST contains two Ss, and one each of R, E, I, and T (6 letters total). Thus, $n_0 = 6!/2!$.

Next compute the number of arrangements in which the combination RT appears. This is the number of ways to arrange the "letters' of the "word" containing two Ss, and one each of RT, E, I (the total of 5). This number equals $5!/2!$.

The number of arrangements containing TR is the same. Thus, $n_1 = 2 \cdot 5!/2! = 5!$, and $n = n_0 - n! = 6!/1 - 5! = 3 \cdot 5! - 5! = 2 \cdot 5!$.

9. Suppose X and Y are sets, $f : X \to Y$ is a map, and A, B are subsets of X. Prove that $f(A) - f(B) \subseteq f(A - B)$.

We have to show that any $y \in f(A) - f(B)$ belongs to $f(A - B)$. Note that y belongs to $f(A)$, but not to $f(B)$. Therefore, there exists (not necessary unique) $x \in A$ s.t. $f(x) = y$, and moreover, $x \notin B$. In other words, $x \in A - B$, hence $y = f(x) \in f(A) - f(B)$.

10. Suppose S is a set of n distinguishable elements. How many ordered pairs (A, B) of subsets of S satisfy $A \subseteq B$?

Place $x \in S$ into one of three bins: Bin 1 if $x \in A$, Bin 2 if $x \in B - A$, and Bin 3 if $x \in S - A$. Different pairs (A, B) correspond to different content of these thee bins. Furthermore, for any distribution of objects into these bins, we can reconstruct a unique pair $A \subseteq B$. Thus, the number of pairs equals to the number of ways to place the objects into three bins. The latter equals 3^n (by Product Rule).

Alternative solution. Denote the total number of pairs by N. Let k_B be the number of pairs (A, B) with fixed B, and N_s the number of pairs with $|B| = s$ $(0 \leq s \leq n)$. Clearly, k_B is the cardinality of the power set of B, that is, $2^{|B|}$. Moreover, the number of B's with cardinality s equals $\binom{n}{s}$, hence $N_s = 2^s \binom{n}{s}$. By Sum Rule,

$$N = \sum_{s=0}^{n} N_s = \sum_{s=0}^{n} \binom{n}{s} 2^s 1^{n-1} = (2+1)^n = 3^n.$$

11. Prove that there exist $m > n$ positive integers $m > n$ so that 2008 divides $2^m - 2^n$.

For $1 \leq n \leq 2008$, write $2^{2009} - 2^n$ as $2008a_n + r_n$, where $a_n \in \mathbb{Z}$ and $r_n \in \{0, \ldots, 2007\}$. If $r_n = 0$ for some n, we are done ($2^{2009} - 2^n$ is a multiple of 2008). Otherwise, $r_n \in \{1, \ldots, 2007\}$ for each $n \in \{1, \ldots, 2008\}$. By the Pigeon-Hole Principle, there exist $m > n$ s.t. $r_m = r_n$. Then

$$2^m - 2^n = (2008a_m + r_m) - (2008a_n + r_n) = 2008(a_m - a_n),$$

which is divisible by 2008.

To: the syllabus, the main page of the course, the problems.