MATH 213: SOLUTIONS FOR MIDTERM 2

1 *(10 points)***:** A biased coin (the probability of getting Heads equals 1/3) is tossed twice. Consider two random variables: let X be the number of Heads obtained, and let Y be equal to 1 if the two tosses yield the same result (both Heads or both Tails), equal to 0 otherwise. Are the random variables X and Y independent?

X is a binomial random variable: $p(X = k) = \binom{2}{k} p^k (1-p)^{2-k}$, hence $p(X = 0) = p(X = 1) = 4/9$, $p(X = 2) = 1/9$. Moreover, $p(Y = 1) = p(HH) + p(TT) = (1/3)^2 + (2/3)^2 = 5/9$, and $p(Y = 0) = 1 - p(Y = 1) = 4/9$. Note now that if $Y = 0$, then $X \leq 1$. Then $0 = p(Y = 0, X = 2) \neq p(Y = 0)p(X = 2) = 4/81$, and the random variables X and Y are **not independent**.

2 *(10 points)***:** The final exam contains 50 true/false questions, each worth 2 points, and 25 multiple-choice questions, each worth 4 points. Linda answers a true/false question correctly with the probability of 0.6. The probability she answers a multiple-choice question correctly is 0.8. Find Linda's expected score.

Denote Linda's score by X. Let X_1 (X_2) be the number of correct answers on the true-false section (resp. multiple-choice section). Then $X = 2X_1 + 4X_2$, hence $\mathbb{E}(X) = 2\mathbb{E}(X_1) + 4\mathbb{E}(X_2)$. Note that X_1 is the number of successes in a series of $n = 50$ independent Bernoulli trials, if the probability of success $p = 0.6$. Thus, X_1 is a binomial random variable, and $\mathbb{E}(X_1) = np = 50 \cdot 0.6 = 30$. Similarly, $\mathbb{E}(X_2) = 25 \cdot 0.8 = 20$. Then $\mathbb{E}(X) = 2 \cdot 30 + 4 \cdot 20 = 140$.

3 *(10 points)***:** What is the probability that a poker hand contains cards of five different kinds?

Recall that a card deck has 52 cards, of 13 different kinds (denominations). There are 4 suits, each containing one card of each denomination. A poker hand consists of 5 cards.

Our sample space S is the set of all selections of 5 cards (our of 52). The total number of poker hands (that is, the number of collections of 5 cards) equals $\binom{52}{5}$. This is the cardinality of our sample space S. The event E occurs if 5 different kinds of cards (out of 13) are present. There are $\binom{13}{5}$ ways to select the kinds of cards. For each of the 5 selected kinds, there are four ways to pick a suit. Thus, $|E| = 4^5 \binom{13}{5}$, and $p(E) = |E|/|S| = 4^5 \binom{13}{5} / \binom{52}{5}$.

4 *(10 points)***:** A sequence (a_n) satisfies the recurrence relation $a_n = 4a_{n-2} - 9n$ for $n \geq 2$, $a_0 = 9$, and $a_1 = 13$. Find an explicit formula for a_n.

$\boldsymbol{a_n = 2^n + 3n + 8}$.

We are dealing with a non-homogeneous linear recurrence relation $a_n = 4a_{n-2} + F(n)$, where $F(n) = -9n$. We shall look for the solution in the form $a_n = a_n^{(h)} + a_n^{(p)}$, where the sequence $(a_n^{(h)})$ solves the homogeneous linear recurrence relation $a_n = 4a_{n-2}$, and $a_n^{(p)}$ is a particular solution to the non-homogeneous recurrence relation. The characteristic equation is $r^2 - 4 = 0$, with roots $r_1 = -2$ and $r_2 = 2$. Thus, $a_n^{(h)} = \alpha_1(-2)^n + \alpha_2 2^n$, for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

As far as $a_n^{(p)}$ is concerned, we have $F(n) = u(n) \cdot 1^n$, where $u(n) = -9n$ is a polynomial of degree 1, and $1 \notin \{r_1, r_2\}$. Thus, we look for $a_n^{(p)} = v(n) \cdot 1^n$, where v is a polynomial of degree 1. In other words, $a_n^{(p)} = \lambda n + \mu$. To find λ and μ, note that

$$\lambda n + \mu = 4(\lambda(n-2) + \mu) - 9n = (4\lambda - 9)n + (4\mu - 8\lambda).$$

Comparing the terms with n, we see that $\lambda = 4\lambda - 9$, hence $\lambda = 3$. The comparison of the constant terms yields $\mu = 4\mu - 8\lambda$, hence $\mu = 8$. Thus, $a_n^{(p)} = 3n + 8$, and the general solution of our recurrence relation is $a_n = \alpha_1(-2)^n + \alpha_2 2^n + 3n + 8$. From the initial condition,

$$\begin{cases} a_0 = 9 = \alpha_1 + \alpha_2 + 8 \\ a_1 = 13 = -2\alpha_1 + 2\alpha_2 + 11 \end{cases}.$$

Solving this system of equations for α_1 and α_2, we obtain the answer.

To: the syllabus, the main page of the course.