

MATH 213 MIDTERM 2: SOLUTIONS FOR PRACTICE PROBLEMS

The test will be given on **Wednesday, November 19**. It will be based on Homeworks 6-10, covering the material from Sections 6.1-7.5 (probability, recurrence relations, and inclusion-exclusion).

In preparing for the test, practice solving the problems from this list. In addition, take a look at the homework (at least one problem on the midterm will come directly from the homework), and at the examples given in the textbook.

1. For each of the following recurrence relations, find an explicit formula for a_n .

(a) $a_n = a_{n-2} - 2a_{n-1}$, $a_0 = 2$, $a_1 = 1$.

(b) $a_n = 2a_{n-1} + 3a_{n-2} + 4(-1)^n$, $a_0 = 1$, $a_1 = -4$.

(a) Update: there is an error in this solution! I was solving $a_n = -a_{n-2} - 2a_{n-1}$. The correct solution is below.

We are dealing with a non-homogeneous linear recurrence relation $a_n = -2a_{n-1} - a_{n-2}$. Its characteristic equation is $r^2 + 2r + 1$ has only one root, namely, -1 . Thus, we have to look for the solution of the form $a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n$. The constants α_1 and α_2 are determined from the initial conditions:

$$\begin{cases} a_0 = 2 = \alpha_1 \\ a_1 = 1 = -\alpha_1 - \alpha_2 \end{cases} .$$

Thus, $\alpha_1 = 2$, and $\alpha_2 = -1$. Therefore, $\mathbf{a_n = (2 - n)(-1)^n}$.

Now, to the corrected version. We are dealing with a non-homogeneous linear recurrence relation $a_n = -2a_{n-1} + a_{n-2}$. Its characteristic equation is $r^2 + 2r - 1 = 0$ has two roots: $r_1 = -1 - \sqrt{2}$ and $r_2 = -1 + \sqrt{2}$. Thus, we look for the solution of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. The constants α_1 and α_2 are determined from the initial conditions:

$$\begin{cases} a_0 = 2 = \alpha_1 + \alpha_2 \\ a_1 = 1 = (-1 - \sqrt{2})\alpha_1 + (-1 + \sqrt{2})\alpha_2 \end{cases} .$$

From the first equation, $\alpha_2 = 2 - \alpha_1$. From the second one,

$$1 = (-1 - \sqrt{2})\alpha_1 + (-1 + \sqrt{2})(2 - \alpha_1) = -2 + 2\sqrt{2} - 2\sqrt{2}\alpha_1.$$

Thus,

$$\alpha_1 = 1 - \frac{3}{2\sqrt{2}} \text{ and } \alpha_2 = 1 + \frac{3}{2\sqrt{2}},$$

hence

$$\mathbf{a_n = \left(1 - \frac{3}{2\sqrt{2}}\right)(-1 - \sqrt{2})^n + \left(1 + \frac{3}{2\sqrt{2}}\right)(-1 + \sqrt{2})^n.}$$

(b) An error in the calculation of α'_1 and α_2 corrected. We are dealing with a non-homogeneous linear recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + F(n)$, where $F(n) = 4(-1)^n$. We shall look for the solution in the form $a_n = a_n^{(h)} + a_n^{(p)}$, where the sequence $(a_n^{(h)})$ solves the homogeneous linear recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$, and $a_n^{(p)}$ is a particular solution to the non-homogeneous recurrence relation. The characteristic equation is $r^2 - 2r - 3 = 0$, with roots $r_1 = -1$ and $r_2 = 3$. Thus, $a_n^{(h)} = \alpha_1(-1)^n + \alpha_2 3^n$, for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

As far as $a_n^{(p)}$ is concerned, we have $F(n) = u(n) \cdot (-1)^n$, where $u(n) = 4$ is a polynomial of degree 0, and $-1 \in \{r_1, r_2\}$. Thus, we look for $a_n^{(p)} = v(n) \cdot (-1)^n$, where v is a polynomial of degree 1. In other words, $a_n^{(p)} = \lambda n + \mu$. To find λ and μ , note that

$$(\lambda n + \mu)(-1)^n = 2(\lambda(n-1) + \mu)(-1)^{n-1} + 3(\lambda(n-2) + \mu)(-1)^{n-2} + 4(-1)^n.$$

Multiplying both sides by $(-1)^n$, we conclude that

$$\lambda n + \mu = -2(\lambda(n-1) + \mu) + 3(\lambda(n-2) + \mu) + 4 = \lambda n + (-4\lambda + \mu + 4),$$

hence $\lambda = 1$. Therefore, $a_n = \alpha'_1(-1)^n + \alpha_2 3^n + n(-1)^n$, where $\alpha'_1 = \alpha_1 + \mu$. The constants α'_1 and α_2 are found from the initial conditions:

$$\begin{cases} a_0 = 1 & = \alpha'_1 + \alpha_2 \\ a_1 = -4 & = -\alpha'_1 + 3\alpha_2 - 1 \end{cases}.$$

Thus, $\alpha'_1 = 3/2$, and $\alpha_2 = -1/2$. Therefore, $a_n = (3(-1)^n - 3^n)/2 + n(-1)^n$.

2. Suppose a random variable X attains only four values $-0, 1, 2,$ and 3 , with $p(X = 0) = 1/2, p(X = 1) = p(X = 2) = p(X = 3) = 1/6$. Compute the expected value and the variance of X .

$$\mathbb{E}(X) = \sum_k k p(X = k) = 0 \cdot \frac{1}{2} + (1 + 2 + 3) \cdot \frac{1}{6} = \mathbf{1}.$$

$$\mathbb{E}(X^2) = \sum_k k^2 p(X = k) = 0^2 \cdot \frac{1}{2} + (1^2 + 2^2 + 3^2) \cdot \frac{1}{6} = \frac{7}{3},$$

hence $V(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbf{4/3}$.

3. A fair coin is tossed four times. Consider the events E and F : E occurs when Heads appears exactly twice, and F occurs when Heads appears exactly once on the first two tosses. Are the events E and F independent?

The sample space S consists of the sequence of four Heads and Tails, hence $|S| = 2^4 = 16$. The event E corresponds to the sequences with exactly two Heads, hence $|E| = \binom{4}{2} = 6$ (the number of ways to select positions for the Heads). F consists of the outcomes where the first two tosses result in either HT, or TH. There are $2 = \binom{2}{1}$ ways to pick the results of the first two tosses, and 2^2 ways to select the result of the last two. Thus, $|F| = 2 \cdot 2^2 = 8$. Finally, $E \cap F$ consists of the outcomes where Heads occurs exactly once on the first two tosses, and once on the last two. Thus, $|E \cap F| = \binom{2}{1}^2 = 4$. Therefore, $p(E) = |E|/|S| = 6/16 = 3/8, p(F) = 8/16 = 1/2,$

and $p(E \cap F) = 4/16 = 1/4$. We see that $p(E)p(F) \neq p(E \cap F)$, hence the events E and F are **not independent**.

4. An urn contains 98 red balls and 2 green balls. Balls are drawn from the urn without repetition, until a green ball is encountered. Denote the number of drawings performed by X .

Compute $p(X = k)$ ($1 \leq k \leq 99$).

Fix $k \in \{1, \dots, 99\}$. Consider the following experiment: balls are drawn from the urn (without replacement) k times. We can assume that the balls are enumerated, with red balls having numbers from 1 to 98, and the green ones, the numbers 99 and 100. The sample space is the set of all sequence of k numbers from 1 to 100 (without repetitions). Thus, $|S| = P(100, k)$. Here, $P(n, r) = n(n-1) \dots (n-r+1) = n!/r!$ is the number of r -permutations of n elements.

$X = k$ iff the first $k-1$ balls bear numbers from 1 to 98, and the k -th one has the number either 99 or 100. There are $P(98, k-1)$ ways to place numbers from 1 to 98 into $k-1$ positions, and 2 ways to pick the number for the k -th position (either 99 or 100). Then $|\{X = k\}| = 2P(98, k-1)$, and

$$p(X = k) = \frac{|\{X = k\}|}{|S|} = \frac{2 \cdot 98 \cdot (100 - k)}{100 \cdot 99 \cdot (101 - k)} = \frac{2(100 - k)}{100 \cdot 99}.$$

5. Among the students taking a certain course, 25% are Biology majors, 25% are Physics majors, and 50% are Math majors (no multiple majors). 80% of Biology majors, 60% of Physics majors, and 40% of Math majors pass this course. What is the probability that a student passing the course is a Math major?

Consider the events E (the student passes the course), F_1 (the student is a Biology major), F_2 (the student is a Physics major), and F_3 (the student is a Math major). We know that

$$p(F_1) = p(F_2) = \frac{1}{4}, \quad p(F_3) = \frac{1}{2}, \quad p(E|F_1) = \frac{4}{5}, \quad p(E|F_2) = \frac{3}{5}, \quad p(E|F_3) = \frac{2}{5},$$

and have to compute $p(F_3|E)$. To achieve this, use Bayes's Formula:

$$p(F_3|E) = \frac{p(E|F_3)p(F_3)}{\sum_{j=1}^3 p(E|F_j)p(F_j)} = \frac{2/5 \cdot 1/2}{4/5 \cdot 1/4 + 3/5 \cdot 1/4 + 2/5 \cdot 1/2} = \frac{4}{11}.$$

6. Suppose the events E and F are such that $p(F) = 1/3$, $p(E|F) = 4/5$, and $p(E|\overline{F}) = 1/5$. Compute $p(E)$.

Note that $p(E) = p(E \cap F) + p(E \cap \overline{F})$. Furthermore, $p(E|F) = p(E \cap F)/p(F)$, hence $p(E \cap F) = p(E|F)p(F) = 4/15$. Furthermore, $p(\overline{F}) = 1 - p(F) = 2/3$, hence $p(E \cap \overline{F}) = p(E|\overline{F})p(\overline{F}) = 2/15$. Therefore, $p(E) = 4/15 + 2/15 = 2/5$.

7. Suppose the events A and B satisfy $p(A|B) < p(A)$. Prove that $p(B|A) < p(B)$.

$p(A|B) = p(A \cap B)/p(B)$, hence $p(A|B) < p(A)$ iff $p(A \cap B) < p(A)p(B)$. If this happens, then also

$$p(B|A) = \frac{p(A \cap B)}{p(A)} < \frac{p(A)p(B)}{p(A)} = p(B).$$

8. The 52 cards of the standard deck are dealt to four players in such a way that each player get 13 cards. What is the probability that each player gets an ace?

The sample space S consists of all ways to distribute 52 objects into 4 distinct boxes, 13 objects per box. Equivalently, it is the number of ways to write a string of numbers 1, 2, 3, 4, of length 52, containing 13 copies of each letter. Thus, $|S| = 52!/(13!)^4$.

We have to compute the probability of the event E s.t. each of the players gets one ace, and 12 other cards. There are $P(4, 4) = 4!$ ways to distribute the four aces among the four players. Furthermore, there are $48!/(12!)^4$ to distribute the remaining 48 cards. Thus, $|E| = 4! \cdot 48!/(12!)^4$, and

$$p(E) = \frac{|E|}{|S|} = \frac{4! \cdot 48! \cdot (13!)^4}{52! \cdot (12!)^4} = \frac{4! \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49}.$$

9. How many integers between 1 and 1000 are divisible by 3 and 7, but not by 5?

Denote the sets of integers between 1 and 1000 divisible by 3, 7, and 5 by A , B , and C , respectively. We have to compute $|(A \cup B) \cap \overline{C}|$. By the Inclusion-Exclusion Principle,

$$(1) \quad |(A \cup B) \cap \overline{C}| = |A \cup B| - |(A \cup B) \cap C|.$$

By the Inclusion-Exclusion Principle again,

$$(2) \quad |A \cup B| = |A| + |B| - |A \cap B| = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{21} \right\rfloor.$$

By a distributive law, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, hence

$$(3) \quad \begin{aligned} |(A \cup B) \cap C| &= |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)| \\ &= |A \cap C| + |B \cap C| - |A \cap B \cap C| = \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{35} \right\rfloor - \left\lfloor \frac{1000}{105} \right\rfloor. \end{aligned}$$

Plugging the right hand sides of (2) and (3) into (1), we obtain:

$$\begin{aligned} |(A \cup B) \cap \overline{C}| &= \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{21} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{35} \right\rfloor + \left\lfloor \frac{1000}{105} \right\rfloor \\ &= 333 + 142 - 47 - 66 - 28 + 9 = \mathbf{343}. \end{aligned}$$

10. Suppose X_1, X_2, \dots, X_8 are Bernoulli random variables, such that $p(X_i = 1) = 1/4$ for each i , and $p(X_i = X_j = 1) = 1/14$ whenever $i \neq j$. Find the expected value and the variance of $X = X_1 + \dots + X_8$.

Hint. Recall that $V(\sum_{i=1}^8 X_i) = \sum_{i=1}^8 V(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$. See the class notes, or Exercise 38 from Section 6.4, for the definition of the covariance.

(i) **THE EXPECTED VALUE.** If Y is a Bernoulli random variable with $p(Y = 1) = p$, then $\mathbb{E}(Y) = p$. By the linearity of the expected value, $\mathbb{E}(X) = \sum_{i=1}^8 \mathbb{E}(X_i) = 8 \cdot 1/4 = 2$.

(ii) **THE VARIANCE.** We know that

$$(4) \quad V\left(\sum_{i=1}^8 X_i\right) = \sum_{i=1}^8 V(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$$

If Y is a Bernoulli random variable with $p(Y = 1) = p$, then $\mathbb{E}(Y) = p(1 - p)$, hence $V(X_i) = 3/16$ for each i . It remains to compute $\text{Cov}(X_i, X_j)$ for $i \neq j$. We know that $\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)$. Furthermore, $\mathbb{E}(X_i) = 1/4$ for any i . Moreover, $X_i X_j$ equals 1 (with the probability $1/14$, as this happens iff $X_i = X_j = 1$) or 0, hence $\mathbb{E}(X_i X_j) = 1 \cdot p(X_i X_j = 1) + 0 \cdot p(X_i X_j = 0) = 1/14$. Therefore, $\text{Cov}(X_i, X_j) = 1/14 - 1/16 = 1/(7 \cdot 16)$. Finally, there are $7 \cdot 8$ distinct ordered pairs (i, j) with $i \neq j$. Thus, by (4),

$$V\left(\sum_{i=1}^8 X_i\right) = 8 \cdot \frac{3}{16} + 7 \cdot 8 \cdot \frac{1}{7 \cdot 16} = \mathbf{2}.$$

11. There are 5 sets, consisting of 100 elements each. Pairwise intersections of these sets contain 25 elements each, the intersection of any three contain 10 elements, and no element belongs to more than three of the five sets. Compute the number of elements in the union of the five sets.

Denote the sets involved by X_1, \dots, X_5 . Then $|X_i| = 100$ for each i , $|X_{i_1} \cap X_{i_2}| = 25$ if $i_1 < i_2$ (there are $\binom{5}{2} = 10$ pairs like this), and $|X_{i_1} \cap X_{i_2} \cap X_{i_3}| = 10$ if $i_1 < i_2 < i_3$ (there are $\binom{5}{3} = 10$ triples like this). If $i_1 < i_2 < \dots < i_k$ for $k \geq 3$, then $X_{i_1} \cap \dots \cap X_{i_k} = \emptyset$. By Inclusion-Exclusion Principle,

$$\begin{aligned} |X_1 \cup \dots \cup X_5| &= \sum_{k=1}^5 (-1)^{k-1} \sum_{i_1 < \dots < i_k} |X_{i_1} \cap \dots \cap X_{i_k}| \\ &= 5 \cdot 100 - \binom{5}{2} 25 + \binom{5}{3} 10 = 500 - 250 + 100 = \mathbf{350}. \end{aligned}$$

To: the syllabus, the main page of the course, the problems.