SOLUTIONS FOR HOMEWORK 1

1.1.12. First show that $f(E) \cup f(F) = f(E \cup F)$. Suppose $b \in B$ belongs to $f(E) \cup f(F)$. Then b belongs to either $f(E)$ or $f(F)$. In the former case, there exists $a \in E$ s.t. $b = f(a)$, while in the latter case, there exists $a \in F$ s.t. $b = f(a)$. Equivalently, there exists $a \in E \cup F$ s.t. $b = f(a)$. The last condition is equivalent to $b \in f(E \cup F)$.

Note that $E \cap F \subseteq E$, hence $f(E \cap F) \subseteq f(E)$. Similarly, $f(E \cap F) \subseteq f(F)$. Therefore, $f(E \cap F) \subseteq f(E) \cap f(F)$.

1.1.15. For $t \in \mathbb{R}$, set $f(t) = (t - a)/(b - a)$. We show that f is a bijection from (a, b) onto $(0, 1)$.

(1) f is injective. Suppose $f(t_1) = f(t_2)$. By the definition of f, this means

$$\frac{t_1 - a}{b - a} = \frac{t_2 - a}{b - a}.$$

Multiplying both sides by $b - a$ and adding a, we see that $t_1 = t_2$, thus establishing injectivity.

(2) f is surjective. We have to show that, for any $y \in (0, 1)$, there exists $x \in (a, b)$ s.t $f(x) = y$, that is, $(x - a)/(b - a) = y$. However, the last equality holds if (and only if) $x = (b - a)y + a$.

1.1.20. (a) Suppose $g \circ f$ is injective. We show that, for $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ iff $a_1 = a_2$ (this is equivalent to the injectivity of f).

If $f(a_1) = f(a_2)$, then $g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2)$. But $g \circ f$ is injective, hence $a_1 = a_2$.

(b) Suppose $g \circ f$ is injective. We show that for any $c \in C$ there exists $b \in B$ s.t. $g(b) = c$.

By assumption, $g \circ f$ is surjective, hence there exists $a \in A$ s.t. $g \circ f(a) = c$. Let $b = f(a)$. Then $g(b) = c$, thus establishing the subjectivity of g.

1.2.5. We use induction to show that, for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n-1} \frac{n(n+1)}{2}. \qquad (A_n)$$

The statement (A_1) is easily verified, forming the base for our induction. The inductive step consists of proving $(A_n) \Rightarrow (A_{n+1})$ for any $n \in \mathbb{N}$. To prove (A_{n+1})

assuming (A_n), write

$$\sum_{k=1}^{n+1}(-1)^{k-1}k^2 = \sum_{k=1}^{n}(-1)^{k-1}k^2 + (-1)^n(n+1)^2 = (-1)^{n-1}\frac{n(n+1)}{2} + (-1)^n(n+1)^2$$
$$= (-1)^n(n+1)^2 - (-1)^n\frac{n(n+1)}{2} = (-1)^n(n+1)\Big((n+1) - \frac{n}{2}\Big)$$
$$= (-1)^n\frac{(n+1)(n+2)}{2},$$

thus establishing (A_{n+1}).

1.2.7. We use induction to prove that $25^n - 1$ is divisible by 8 for any $n \in \mathbb{N}$. The base of induction is easy to establish by considering $n = 1$. To handle the inductive step, suppose $25^n - 1$ is divisible by 8, and show that $25^{n+1} - 1$ is divisible by 8. Note that

$$(25^{n+1} - 1) - (25^n - 1) = 25^{n+1} - 25^n = 24 \cdot 25^n,$$

hence $(25^{n+1} - 1) = (25^n - 1) + 24 \cdot 25^n$. By the induction hypothesis, both terms on the right are divisible by 8. Therefore, The left hand side is also divisible by 8.

Alternative solution. We know that, for any $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$, $a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$. In particular,

$$25^n - 1 = 25^n - 1^n = (25-1)\cdot\sum_{k=0}^{n-1} 25^k 1^{n-1-k} = 24 \cdot \sum_{k=0}^{n-1} 25^k,$$

which is clearly divisible by 8.

1.2.15. We prove by induction that

$$2^{n-1} - (2n-3) \geq 0 \qquad (B_n)$$

for any $n \in \mathbb{N}$. It is easy to verify B_n for $n = 1, 2$. This is the base of induction.

To handle the inductive step, we need to prove the implication $B_n \Rightarrow B_{n+1}$ for $n \geq 2$. To achieve this, it suffices to show that, for any $n \geq 2$,

$$2^n - (2n-1) \geq 2^{n-1} - (2n-3).$$

This is equivalent to showing that

$$2^n - 2^{n-1} \geq (2n-1) - (2n-3). \qquad (*)$$

The right hand side above equals 2, while the left hand side equals 2^{n-1}. However, $n - 1 \geq 1$, hence $2^{n-1} \geq 2^1 = 2$. This proves (*).

Bonus problem A (*very little partial credit is given*): Suppose we have n lines in a plane, in such a way that no two lines are parallel, and no three lines have a common point. Prove that these lines divide the plane into $(n^2 + n + 2)/2$ regions.

We say that n lines in a plane are *in the general position* if no two lines are parallel, and no three lines have a common point.

The proof proceeds by induction. The case of $n = 1$ is easy: 1 line divides the plane into $2 = (1^2 + 1 + 2)/2$ regions (you can check the formula for $n = 2$ in the same manner).

Now, suppose we have n lines in the general position, which divide the plane into K regions: $A_1, \ldots, A_K$. Add one more line (call it L) in such a way that the $n+1$ lines are still in the general position. Theses $n + 1$ lines give rise to M regions. We show that $M = K + n + 1$.

Note that $B - A$ equals the number of regions A_i intersected by L (compare the regions divided by L into two, versus those it leaves unaltered). By rotating the plane, we can assume that L coincides with the x-axis. Let's travel along L from $-\infty$ to ∞. To the left of a certain point, L lies in the region A_{i_0}. After meeting one line, it passes to A_{i_1}. After intersecting the second line, L moves into A_{i_2}. Finally, to the right of its intersection with the n-th line, L lies in a region A_n.

So, the grand total of the regions intersected by L equals $n + 1$.

We now complete the inductive step. Suppose the number of regions equals $(n^2 + n + 2)/2$ for n lines, and prove that, for $n + 1$ lines, it must be equal to $((n + 1)^2 + (n + 1) + 2)/2$. By the above, for $n + 1$ lines

$$\begin{aligned}\#\,\text{of regions} &= \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2} \\ &= \frac{(n^2 + 2n + 1) + (n + 1) + 2}{2} = \frac{(n + 1)^2 + (n + 1) + 2}{2},\end{aligned}$$

as desired.

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