SOLUTIONS FOR HOMEWORK 10

6.2.2. (b) Let $g(x) = x/(x^2+1)$. By the quotient rule, $g'(x) = (1-x^2)/(x^2+1)^2$. Thus, $g'(x) \geq 0$ iff $x \in [-1, 1]$, and $g'(x) \leq 0$ iff $x \in (-\infty, -1] \cup [1, \infty)$. Therefore, g increases on $[-1, 1]$, and decreases on the intervals $(-\infty, -1]$ and $[1, \infty)$.

6.2.3. (b) $g(x) = 1 - (x-1)^{2/3}$, $0 \leq x \leq 2$. Note that g is continuous. For $x \neq 0$, $g'(x) = -2(x-1)^{-1/3}/3$. g' is positive for $x < 1$, and negative for $x > 1$. Thus, g increases on $[0, 1]$, decreases on $[1, 2]$, and has a relative maximum (its only relative extremum) at 1.

6.2.5. For $x \geq 1$, let $f(x) = x^{1/n} - (x-1)^{1/n}$. Note that f is continuous on $[1, \infty)$, and differentiable on $(1, \infty)$. Moreover,

$$f'(x) = \frac{1}{n}\Big(\frac{1}{x^{(n-1)/n}} - \frac{1}{(x-1)^{(n-1)/n}}\Big) < 0,$$

hence f is strictly decreasing on $[1, \infty)$. Therefore,

$$f(1) = 1 > f(a/b) = \frac{a^{1/n} - (a-b)^{1/n}}{b^{1/n}}.$$

Multiplying both sides by $b^{1/n}$, we obtain $b^{1/n} > a^{1/n} - (a-b)^{1/n}$, hence $(a-b)^{1/n} > a^{1/n} - b^{1/n}$.

6.2.7. Let $f(x) = \ln x$. Then $g(x) = f'(x) = 1/x$ is a strictly decreasing function on $(0, \infty)$. By Mean Value Theorem, for any $x > 1$ there exists $c \in (1, x)$ s.t. $f'(c)(x-1) = f(x) - f(1)$. But

$$f'(x) = \frac{1}{x} < f'(c) < 1 = f'(1),$$

hence (recalling that $f(1) = \ln 1 = 0$)

$$\frac{x-1}{x} < \ln x < x - 1.$$

6.2.10. Write $g = g_1 = g_2$, where $g_1(x) = x$, and

$$g_2(x) = \begin{cases} 0 & x = 0 \\ 2x^2 \sin(1/x) & x \neq 0 \end{cases}.$$

Clearly, $g_1' = 1$ everywhere, and $g_2'(x) = 4x \in (1/x) - 2\cos(1/x)$. We next show that $g_2'(0) = 0$. Indeed, $|g_2(x) - g_2(0)| \leq 2x^2$ for any x, hence $|g_2(x) - g_2(0)|/|x - 0| \leq 2|x|$ for $|x| \neq 0$. Therefore, for any $\varepsilon > 0$,

$$\Big|\frac{g_2(x) - g_2(0)}{x - 0} - 0\Big| < \varepsilon$$

whenever $0 < |x| < \varepsilon/2$, which implies $g_2'(0) = 0$.

We conclude that g is differentiable everywhere, with

$$g'(x) = g_1'(x) + g_2'(x) = \begin{cases} 1 & x = 0 \\ 1 + 4x \in (1/x) - 2\cos(1/x) & x \neq 0 \end{cases}.$$

In particular, $g'(0) > 1$. Now let $x_n = 1/((2n+1)\pi)$. Then $g'(x_n) = 1 + 0 - 2 = -1 < 0$, and $\lim x_n = 0$. Therefore, there is no $\delta > 0$ s.t. g increases on $(-\delta, \delta)$. Indeed, if such a δ existed, then we would have $g'(t) \geq 0$ for any $t \in (-\delta, \delta)$, which is impossible.

6.2.11. Let $f(x) = \sqrt{x}$. Then f is continuous, hence uniformly continuous, on $[0, 1]$. However, $f'(x) = x^{-1/2}/2$ for $x > 0$. This function is unbounded: for any $M > 0$ there exists $x \in (0, 1)$ s.t. $f'(x) > M$.

6.2.14. Suppose, for the sake of contradiction, that f changes sign on I. That is, there exist $a < b$ in the interior of I s.t. $f'(a)f'(b) < 0$. By Darboux's Theorem (6.2.12), there exists $c \in (a, b)$ s.t. $f'(c) = 0$.

6.2.15. There exists $K > 0$ is such that $|f'(x)| \leq K$ for any $x \in K$. Consider $a, b \in I$, $a < b$. By Mean Value Theorem, there exists $c \in (a, b)$ s.t. $f'(c)(b-a) = f(b) - f(a)$. Therefore, $|f(b) - f(a)| \leq K|b - a|$.

6.2.19. *This is a bonus problem – very little partial credit is given.* Suppose f is uniformly differentiable on I, and show that it is uniformly continuous. To this end, we have to prove that, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f'(x) - f'(y)| < \varepsilon$ whenever $|y - x| < \delta$. Select δ in such a way that

$$\Big|\frac{f(x) - f(y)}{x - y} - f'(x)\Big| < \frac{\varepsilon}{2}$$

whenever $|x - y| < \delta$. Interchanging the roles of x and y in this inequality, we see that, for such x and y,

$$\Big|\frac{f(x) - f(y)}{x - y} - f'(y)\Big| < \frac{\varepsilon}{2},$$

hence, by the triangle inequality,

$$|f'(x) - f'(y)| \leq \Big|\frac{f(x) - f(y)}{x - y} - f'(x)\Big| + \Big|\frac{f(x) - f(y)}{x - y} - f'(y)\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is what we need.

6.2.20. **(a)** Apply Mean Value Theorem to $[0, 1]$ to obtain $c_1 \in (0, 1)$ s.t. $f'(c_1) = (f(1) - f(0))/(1 - 0) = 1$.

(b) Apply Mean Value Theorem to $[1, 2]$ to obtain $c_2 \in (1, 2)$ s.t. $f'(c_2) = (f(2) - f(1))/(2 - 1) = 0$.

(c) Apply Mean Value Theorem for Derivatives (Darboux's Theorem) to $[c_1, c_2]$ to obtain $c \in (c_1, c_2)$ s.t. $f'(c) = 1/3$ (clearly, $1/3 \in (0, 1)$).

Back to: the syllabus, the main page of the course, the assignment.