SOLUTIONS FOR HOMEWORK 11

7.1.5. Suppose $\mathcal{P} = (a = x_0 < x_1 < \ldots < x_n = b)$ is a partition, with $I_k = [x_{k-1}, x_k]$, and $\dot{\mathcal{P}}$ is the same partition equipped with tags $t_k \in I_k$.

(a) Suppose $u \in I_k = [x_{k-1}, x_k]$, and $t_k \in [c_1, c_2]$. Then $x_{k-1} \le t_k \le c_2$, hence

$$x_k = x_{k-1} + (x_k - x_{k-1}) \le x_{k-1} + \|\mathcal{P}\| \le c_2 + \|\mathcal{P}\|,$$

and similarly, $x_{k-1} \ge c_1 - \|\mathcal{P}\|$. Then $x_1 - \|\mathcal{P}\| \le x_{k-1} \le u \le x_k \le c_2 + \|\mathcal{P}\|$.

(b) Suppose $v \in I_k = [x_{k-1}, x_k]$ satisfies $c_1 + \|\mathcal{P}\| \le c_2 - \|\mathcal{P}\|$. Then

$$t_k \le x_k \le x_{k-1} + \|\mathcal{P}\| \le v + \|\mathcal{P}\| = c_2,$$

and similar,y $t_k \ge c_1$.

7.1.9. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ s.t. $|S(f, \dot{\mathcal{P}}) - \int_a^b f| < \varepsilon$ whenever $\|\dot{\mathcal{P}}\| < \delta$. Moreover, there exists $N \in \mathbb{N}$ s.t. $\|\mathcal{P}_n\| < \delta$ for any $n > N$. Therefore, $|S(f, \dot{\mathcal{P}}_n) - \int_a^b f| < \varepsilon$ for any $n > N$. Thus, $\lim S(f, \dot{\mathcal{P}}_n) = \int_a^b f$.

7.1.13. Suppose f differs from 0 only at the points $c_1 < c_2 < \ldots c_m$. Let $M = \max_{1 \le i \le m} |f(c_i)|$. We shall show that, for any tagged partition $\dot{\mathcal{P}}$, $|S(f, \dot{\mathcal{P}})| \le 2Mm\|P\|$. Once this is established, we will be done: for any $\varepsilon > 0$, and any tagged partition $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \varepsilon/(2Mm)$, we would have $|S(f, \dot{\mathcal{P}})| < \varepsilon$.

For a partition $\mathcal{P} = (a = x_0 < x_1 < \ldots < x_n = b)$, let $\mathcal{I}$ be the set of all $k \in \{1, \ldots, n\}$ s.t. $c_i \in I_k = [x_{k-1}, x_k]$ for some (perhaps more than one) i. Note that $f(t_j) = 0$ if $j \notin \mathcal{I}$, and $|f(t_j)| \le M$ for $j \in \mathcal{I}$. Furthermore, if $c_i \in [x_{k-1}, x_k]$, then $c_i - \|\mathcal{P}\| \le x_{k-1} < x_k \le c_i + \|\mathcal{P}\|$. This implies

$$\sum_{j \in \mathcal{I}} (x_j - x_{j-1}) \le \sum_{i=1}^m \sum_{c_i \in I_j} (x_j - x_{j-1}) \le \sum_{i=1}^m 2\|P\| = 2m\|P\|.$$

Therefore,

$$|S(f, \dot{\mathcal{P}})| \le \sum_{j \in \mathcal{I}} |f(t_j)|(x_j - x_{j-1}) \le M \sum_{j \in \mathcal{I}} (x_j - x_{j-1}) \le 2Mm,$$

which is what we want.

7.1.14. Let $h = f - g$. Then h differs from 0 at finitely many points, hence $h \in \mathcal{R}[a, b]$, and $\int_a^b h = 0$. Therefore, $f = g + h \in \mathcal{R}[a, b]$, and $\int_a^b f = \int_a^b g + \int_a^b h = \int_a^b g$.

7.1.15. As suggested in the textbook, let $\delta = \alpha/(4\varepsilon)$. Suppose a partition $\mathcal{P} = (a = x_0 < x_1 < \ldots < x_n = b)$ satisfies $\|\mathcal{P}\| < \delta$. We shall show that

$$\alpha(d - c - 2\delta) \le S(\phi, \dot{\mathcal{P}}) \le \alpha(d - c + 2\delta) \qquad (1)$$

Indeed, the above inequality implies $|S(\phi, \dot{\mathcal{P}}) - \alpha(d-c)| \leq \varepsilon/2$ whenever $\|\dot{\mathcal{P}}\| \leq \delta$, which implies $\int_a^b \phi = \alpha(d-c)$.

First consider the easy case: there is no k s.t. $x_k \in [c,d]$. Then $[c,d] \subset (x_{j-1}, x_j)$ for some j, hence (depending on whether t_j belong to $[c,d]$ or not), either $S(\phi, \dot{\mathcal{P}}) = \alpha(x_j - x_{j-1})$, or $S(\phi, \dot{\mathcal{P}}) = 0$. Then $d - c < \delta$, hence

$$\alpha(d-c-2\delta) < 0 \leq S(\phi, \dot{\mathcal{P}}) \leq \alpha(x_j - x_{j-1}) \leq \alpha\delta \leq \alpha(d-c+2\delta),$$

which establishes (1).

Next deal with the harder case: there is a k s.t. $x_k \in [c,d]$. Denote by i_0 the smallest i s.t. $x_i \geq c$, and by i_1 the smallest i s.t. $x_i > d$. Then $f(t_i) = 0$ if $i < i_0$ or $i > i_1$, and $f(t_i) = \alpha$ if $i_0 + 1 \leq i \leq i_1 - 1$. $f(t_{i_0})$ and $f(t_{i_1})$ can be either 0 or α. Thus, the smallest value of $S(\phi, \dot{\mathcal{P}})$ equals $\sum_{i=i_0+1}^{i_1-1} \alpha(x_i - x_{i-1}) = \alpha(x_{i_1-1} - x_{i_0})$. But $x_{i_0} \leq c + \delta$, and $x_{i_1-1} \geq d + \delta$, hence $S(\phi, \dot{\mathcal{P}}) \geq \alpha(d-c-2\delta)$, thus establishing the left hand side of (1). Similarly,

$$S(\phi, \dot{\mathcal{P}}) \leq \sum_{i=i_0}^{i_1} \alpha(x_i - x_{i-1}) \leq \alpha(d-c+2\delta),$$

which establishes the right hand side of (1).

Alternatively, one can combine the additivity of integrals (Theorem 7.2.8) with Exercise 7.1.13.

7.2.12. Note that $|f(x)| \leq 1$ for any $x \in [0,1]$. Moreover, the restriction of f to $[\varepsilon, 1]$ is continuous for any $\varepsilon > 0$ (as a composition of two continuous functions). Therefore, $f \in R[\varepsilon, 1]$. An application of Exercise 11 (proved in class) completes the proof.

7.2.17. Let m and M be the minimal and maximal values of f on $[a,b]$ (by the continuity of f, they exist, and are attained). For $a \leq t \leq b$, consider the function $F(t) = t\int_a^b g - \int_a^b fg = \int_a^b (t - f(x))g(x)\,dx$. The function g is non-negative, hence $(M - f(x))g(x) \geq 0$ for any $x \in [a,b]$, Therefore, $F(M) \geq 0$. Similarly, we show that $F(m) \leq 0$. Moreover, F is continuous (in fact, it is a polynomial of degree 1), hence, by Intermediate Value Theorem, there exists $y \in [m, M]$ s.t. $F(y) = 0$, or in other words, $y\int_a^b g = \int_a^b fg$. The function f attains both $m \leq y$ and $M \geq y$, hence, by Intermediate Value Theorem again, there exists $c \in [a,b]$ s.t. $f(c) = y$.

To provide a counterexample for the case when the requirement $g \geq 0$ is omitted, consider $[a,b] = [-2,2]$,

$$f(x) = \begin{cases} -1-x & -2 \leq x \leq -1 \\ 0 & -1 < x \leq 2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} -1 & -2 \leq x \leq -1 \\ x & -1 < x \leq 2 \end{cases}.$$

We showed in class that $\int_0^1 x\,dx = 1/2$. In the same way (or using Fundamental Theorem of Calculus), one can show that $\int_a^b x\,dx = (b^2 - a^2)/2$. Then

$$\int_{-2}^{2} g = \int_{-2}^{-1} (-1)\,dx + \int_{-1}^{2} x\,dx = (-1) + \frac{2^2 - (-1)^2}{2} = \frac{1}{2}.$$

However,

$$f(x)g(x) = \begin{cases} 1+x & -2 \le x \le -1 \\ 0 & -1 < x \le 2 \end{cases},$$

hence $\int_{-2}^{2} fg = \int_{-2}^{-1} (x+1)\, dx = -1/2$. However, f is non-negative everywhere, hence there is no $c \in [-2, 2]$ s.t. $-1/2 = f(c) \cdot 1/2$.

Alternatively, one could have used $f(x) = g(x) = x$ on $[-1, 1]$. Then $\int_{-1}^{1} g = 0$. On the other hand, $[x^3/3]' = x^2$, hence, by Fundamental Theorem of Calculus, $\int_{-1}^{1} fg = \int_{-1}^{1} x^2\, dx = (1^3 - (-1)^3)/3 = 2/3$. Clearly, there is no c s.t. $2/3 = f(c) \cdot 0$.

7.2.18. *This is a bonus problem – very little partial credit is given.* Let $M = \sup\{f(x) : a \le x \le b\}$, and $M_n = \big(\int_a^b f^n\big)^{1/n}$. Clearly, $M_n^n = \int_a^b f^n \le M^n$, hence it suffices to show that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $M_n > M - \varepsilon$ for every $n > N$.

The function f is continuous, hence there exists $c \in [a, b]$ s.t. $f(c) = M$. Fix $\varepsilon > 0$. Then there exists $\delta \in (0, (b-a)/2)$ s.t. $|f(x) - f(c)| < M\varepsilon/2$ for any $x \in [a, b] \cap [c - \delta, c + \delta]$. Let $\alpha = \max\{a, c - \delta\}$, and $\beta = \min\{c + \delta, b\}$. As either $c - \delta$ or $c + \delta$ belongs to $[a, b]$, $\beta - \alpha \ge \delta$. Note that $f(x) > M(1 - \varepsilon/2)$ for any $x \in [\alpha, \beta]$. By Corollary 7.2.10,

$$\begin{aligned} M_n^n = \int_a^b f^n &= \int_a^\alpha f^n + \int_\alpha^\beta f^n + \int_\beta^b f^n \\ &\ge \int_\alpha^\beta f^n \ge (\beta - \alpha) M^n \Big(1 - \frac{\varepsilon}{2}\Big)^n \ge \delta M^n \Big(1 - \frac{\varepsilon}{2}\Big)^n, \end{aligned}$$

hence $M_n \ge \delta^{1/n} M(1 - \varepsilon/2)$. By Example 3.1.11(b), $\lim \delta^{1/n} = 1$, hence there exists $N \in \mathbb{N}$ s.t. $\delta^{1/n} > (1 - \varepsilon)/(1 - \varepsilon/2)$ for any $n > N$. For such n, $M_n > (1 - \varepsilon)M$, as desired.

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