SOLUTIONS FOR HOMEWORK 2

1.3.8. For $k \in \mathbb{N}$ let $A_k = \{1, 2, \ldots, k\}$. Then $|A_k| = k$, yet $\cup_{k=1}^{\infty} A_k = \mathbb{N}$.

1.3.12. Let $\mathcal{F}_m$ be the family of subsets of $\mathbb{N}$ with precisely m elements. Let $\mathbb{N}^m = \mathbb{N} \times \ldots \times \mathbb{N}$ (m times) be the m-fold Cartesian product of $\mathbb{N}$ with itself, that is, the set of all ordered m-tuples whose elements are positive integers. Use induction on m to prove that $\mathbb{N}^m$ is countable. The case of $m = 1$ is clear. The case of $m = 2$ was proved in class: there exists a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. This forms the base of induction. Now, suppose $\mathbb{N}^m$ is countable, and prove the same for $\mathbb{N}^{m+1}$. By the induction hypothesis, there exists an injection $f_m : \mathbb{N}^m \to \mathbb{N}$. For $a = (a_1, \ldots, a_m, a_{m+1}) \in \mathbb{N}^{m+1}$, let $a' = (a_1, \ldots, a_m)$. Define f_{m+1} by setting $f_{m+1}(a) = f_2(f_m(a', a_{m+1}))$. You can check that f_{m+1} is injective, hence $\mathbb{N}^m$ is countable.

Next we define a map $g : \mathcal{F}_m \to \mathbb{N}^m$, taking a set of m elements to the m-tuple, listing them in the increasing order. This is an injection, hence so is $f_{m+1} \circ g$. Therefore, $\mathcal{F}_m$ is countable.

Finally, $\mathcal{F} = \emptyset \cup (\cup_{m=1}^{\infty} \mathcal{F}_m)$, hence countable (as a countable union of countable sets.

2.1.1. **(b)** First make an important observation: the negative element is unique. More precisely:

Lemma. *If $a, b \in \mathbb{R}$, and $a + b = 0$, then $b = -a$.*

Proof. If $a + b = 0$, then $b + a = 0$ (commutativity of addition), hence

$$-a = 0 + (-a) = (b + a) + (-a) = b + (a + (-a)) = b + 0 = b,$$

which is what we need. ■

Thus, for $b \in \mathbb{R}$, $-b$ is the **unique** $c \in \mathbb{R}$ satisfying $b + c = 0$.

In particular, $-(-a)$ is the unique real number with the property that $(-a) + (-(-a)) = 0$. But $(-a) + a = 0$, hence $a = -(-a)$.

2.1.2. **(b)** As in the previous problem, start with

Lemma. *If $a, b \in \mathbb{R}$, then $(-a)b = -ab$.*

Proof. By the distributive law, $ab + (-a)b = (a + (-a))b = 0 \cdot b = 0$, hence, by the previous lemma, $(-a)b = -ab$. ■

Therefore,

$$(-a)(-b) = -a(-b) = -((-b)a) = -(-ba) = ba = ab.$$

Problem A: Denote by $\mathcal{S}$ the set of all subsets $A \subset \mathbb{N}$ for which both A and $\mathbb{N} \backslash A$ are infinite. Prove that $\mathcal{S}$ is uncountable.

We can wrote $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) as a union of three disjoint sets: $\mathcal{S}$, $\mathcal{F}$, and $\mathcal{F}'$, the latter being the set of all those subsets of $\mathbb{N}$ whose complements belong to $\mathcal{F}$.

Above, we established that $\mathcal{F}$ is countable. The operation $A \mapsto \mathbb{N}\backslash A$ is a bijection from $\mathcal{F}$ to $\mathcal{F}'$, hence $\mathcal{F}'$ is also countable. If $\mathcal{S}$ is countable, then so is $\mathcal{P}(\mathbb{N})$, which is impossible by Cantor's theorem (1.3.13).

Problem B: Suppose A is an infinite set, and B is a countable set. Prove that the sets A and $A \cup B$ are equipollent.

Note that, by considering $B\backslash A$ instead of B, we can assume that A and B are disjoint. We consider the case of B being countably infinite (the finite case is handled in a similar fashion). There exists a bijection $f : \mathbb{N} \to B$. Moreover, as shown in the notes, $|A| \geq |\mathbb{N}|$, hence there exists an injection $g : \mathbb{N} \to A$. Now define $h : A \cup B \to A$ by setting

$$h(x) = \begin{cases} x & x \in A\backslash g(\mathbb{N}) \\ g(2k) & x = g(k) \text{ for some } k \in \mathbb{N} \\ g(2k-1) & x = f(k) \text{ for some } k \in \mathbb{N} \end{cases} .$$

Clearly, h is a bijection.

Problem C (*a bonus problem – very little partial credit is given*): For a set A, denote by $A^{\mathbb{N}}$ the set of all infinite sequences of elements of A. Show that $\mathbb{R}^{\mathbb{N}}$ (the set of all sequences of real numbers) is equipollent with $\mathbb{R}$. You can use the fact that $\mathbb{R}$ is equipollent with the power set of $\mathbb{N}$, and identify the latter with the set of all $0-1$ sequences.

By Cantor-Bernstein-Schroeder Theorem, we have to prove the existence of injections $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, and $g : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. f is easy to construct: define $f(x) = (x, x, \ldots)$. To construct g, recall that there exists a bijection $\phi : \mathbb{R} \to \{0,1\}^{\mathbb{N}}$. Denote by $\phi_i(x)$ the i-th term of the sequence $\phi(x)$.

There exists a bijection $\psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Define π and σ by setting $\psi(a) = (\pi(a), \sigma(a))$ (π and σ map $\mathbb{N}$ to itself).

For $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, define $g(x)$ as a $0-1$ sequence, whose m-th element equals $\psi_{\pi(m)}(x_{\sigma(m)})$. Check that g is an injection!

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