SOLUTIONS FOR HOMEWORK 3

2.1.9. **(a)** Suppose $x_1 = s_1 + t_1\sqrt{2}$ and $x_2 = s_2 + t_2\sqrt{2}$, with $t_1, t_2 \in \mathbb{Q}$. Then $x_1 + x_2 = (s_1 + s_2) + (t_1 + t_2)\sqrt{2}$. As sums of rational numbers are rational, $x_1 + x_2 \in K$. Moreover, $x_1 x_2 = (s_1 s_2 + 2t_1 t_2) + (s_1 t_2 + s_2 t_1)\sqrt{2}$. Products and sums of rational numbers are rational, hence $x_1 x_2 \in K$.

(b) Note that $s + t\sqrt{2} = 0$ if and only if $s = t = 0$. Indeed, if $s + t\sqrt{2} = 0$, then $s = -t\sqrt{2}$, hence $s^2 = 2t^2$. If $s = 0$, then $t = 0$, and vice versa. If neither s nor t equals 0, then $2 = r^2$, where $r = s/t$ is a rational number. This, however, is impossible.

So, consider $s + t\sqrt{2}$ with s and t different from 0. Then

$$\frac{1}{s+t\sqrt{2}} = \frac{1}{s+t\sqrt{2}} \cdot \frac{1}{s-t\sqrt{2}} \cdot (s - t\sqrt{2}) = \frac{1}{(s+t\sqrt{2})(s-t\sqrt{2})} \cdot (s - t\sqrt{2})$$
$$= \frac{s-t\sqrt{2}}{s^2-2t^2} = \frac{s}{s^2-2t^2} - \frac{t}{s^2-2t^2}\sqrt{2} \in K,$$

since the ratio of two rational numbers is again rational.

2.2.15. Let $\varepsilon = |b-a|/3$. Then $V_\varepsilon(a) \cap V_\varepsilon(b) = \emptyset$. Indeed, suppose, for the sake of contradiction, that $c \in V_\varepsilon(a) \cap V_\varepsilon(b)$. Then $|a-c| < \varepsilon$ and $|b-c| < \varepsilon$. Therefore, by the triangle inequality (or, more precisely, by Corollary 2.2.4),

$$3\varepsilon = |b-a| = |(b-c)-(a-c)| \le |b-c| + |a-c| < 2\varepsilon,$$

which yields a contradiction.

2.2.16. **(a)** Up to re-labeling, we can assume that $a \le b$. Then $|a-b| = b-a$. Therefore,

$$\frac{a+b+|a-b|}{2} = \frac{a+b+(b-a)}{2} = b = \max\{a,b\},$$
$$\frac{a+b-|a-b|}{2} = \frac{a+b-(b-a)}{2} = a = \min\{a,b\}.$$

2.3.6. Suppose $u \in S$ is an upper bound for $S \subset \mathbb{R}$. Then $v = \sup S$ exists (S is bounded from above), and $v \ge u$ (since $u \in S$). On the other hand, u is an upper bound for S, hence $u \ge v$. Therefore, $v = u$.

2.3.9. Let $a = \sup A$ and $b = \sup B$. Let $c = \max\{a,b\}$. By Exercise 2.3.6, c is the supremum of the set $\{a,b\}$. c is an upper bound for $A \cup B$. Indeed, if $x \in A \cup B$, then either $x \in A$, and $x \le a \le c$, or $x \in B$, and $x \le b \le c$. On the other hand, suppose, for the sake of contradiction, that $d < c$ is an upper bound for $A \cup B$. By relabeling if necessary, we can assume $c = a$. By Lemma 2.3.3, there exists $x \in A$ s.t. $d < x < a$. Then $x \in A \cup B$, hence d is not an upper bound for $A \cup B$.

2.4.2. $S = \{1/m - 1/n : m, n \in \mathbb{N}\}$. We claim that $\inf S = -1$ and $\sup S = 1$. We prove the equality concerning the supremum, as the infimum is handled similarly. Note that $m \geq 1$ for any m, hence $0 < 1/m \leq 1$, and $1/m - 1/n \leq 1/m \leq 1$. Therefore, 1 is an upper bound for S. Now suppose u is another upper bound for S. Taking $m = 1$, we see that $1 - 1/n \leq u$ for any $n \in \mathbb{N}$, or in other words, $1 - u \leq 1/n$ for any such $\mathbb{N}$. If $u < 1$, then, by Exercise 2.1.15, $n < 1/(1-u)$ for every n, which contradicts the Archimedian Property (2.4.3).

2.4.16. Observe (for later use) that, for positive numbers x and y, $x < y$ iff $x^3 < y^3$. Indeed, $x^3 - y^3 = (x-y)(x^2+xy+y^2)$. But $x^2 + xy + y^2 > 0$ for positive x and y, hence $x - y$ and $x^3 - y^3$ have the same sign.

In fact, $x^2+xy+y^2 = \big(x^2+y^2+(x+y)^2\big)/2$ is non-negative for all values of x and y, and equals 0 iff $x = y = 0$. Hence, $x < y$ iff $x^3 < y^3$ for any real x and y. However, in this exercise, we restrict our attention to positive numbers.

Let $S = \{x \in \mathbb{R} : x > 0,\ x^3 \leq 2\}$. The set S is non-empty ($1 \in S$). Moreover, S is bounded above. In fact, $x < 2$ for any $x \in S$. Indeed, if $x \geq 2$, then $x^3 = x \cdot x \cdot x \geq 8 > 2$.

Let $u = \sup S$. Clearly, $1 \leq u \leq 2$. We shall show that $u^3 = 2$.

Suppose first $u^3 > 2$. Find $n \in \mathbb{N}$ s.t. $n > 13/(u^3 - 2)$. Then

$$(u - 1/n)^3 = u^3 - 3u^2/n + 3u/n^2 - 1/n^3 > u^3 - (3u^2+1)/n \geq u^3 - 13/n > 2.$$

Therefore, $(u - 1/n)^3 > x^3$ whenever $0 < x^3 \leq 2$, hence $u - 1/n$ is an upper bound for S, which contradicts our assumption that u is the *least* upper bound.

Now suppose $u^3 < 2$. Find $n \in \mathbb{N}$ s.t. $n > 19/(2 - u^3)$. Then

$$\left(u + \frac{1}{n}\right)^3 = u^3 + \frac{3u^2}{n} + \frac{3u}{n^2} + \frac{1}{n^3} \leq u^3 + \frac{3u^2+3u+1}{n} \leq u^3 + \frac{19}{n} < 2.$$

Thus, $u + 1/n \in S$, hence u is not an upper bound for S.

Problem A: Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Let $x = \sqrt{2} + \sqrt{3}$. If x is rational, then so is x^2. However, $x^2 = 5 + 2\sqrt{6}$. If x^2 is rational, then so is $\sqrt{6} = (x^2 - 5)/2$. However, 6 is not a complete square, hence $\sqrt{6}$ is irrational. Therefore, x cannot be rational.

Problem B (*a bonus problem – very little partial credit is given*): Prove that $|\sqrt{2} - m/n| \geq 1/(8n^2)$ for any $m, n \in \mathbb{N}$.

Suppose, for the sake of contradiction, $|\sqrt{2} - m/n| < 1/(8n^2)$.

Consider first the case of $m/n > \sqrt{2}$. Equivalently, $m^2 > 2n^2$. Then $m^2 \geq 2n^2 + 1$, or in other words, $m^2/n^2 \geq 2 + 1/n^2$. However, if $|\sqrt{2} - m/n| < 1/(8n^2)$, then $m/n < \sqrt{2} + 1/(8n^2)$, hence

$$\frac{m^2}{n^2} < \left(\sqrt{2} + \frac{1}{8n^2}\right)^2 = 2 + \frac{1}{2\sqrt{2}n^2} + \frac{1}{64n^4} < 2 + \frac{1}{n^2},$$

a contradiction.

Now suppose $m/n < \sqrt{2}$. Then $m^2 \leq 2n^2 - 1$, hence $m^2/n^2 \leq 2 - 1/n^2$. On the other hand, $m/n > \sqrt{2} + 1/(8n^2)$, hence $m^2/n^2 > 2 - 1/n^2$, yielding a contradiction.

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