SOLUTIONS FOR HOMEWORK 4

3.1.5. (b) Pick $\varepsilon > 0$, and show that there exists $N \in \mathbb{N}$ s.t. $|2n/(n+1) - 2| < \varepsilon$ for $n > N$. A simple calculation shows that $|2n/(n+1) - 2| = 2/(n+1)$. We know that there exists $N \in \mathbb{N}$ s.t. $1/N < \varepsilon/2$. Then $|2n/(n+1) - 2| < \varepsilon$ for $n > N$.

3.1.10. Let $\varepsilon = x/2$. Then there exists $M \in \mathbb{N}$ s.t. $|x - x_n| > \varepsilon$ for any $n > M$. By the triangle inequality, for such n

$$x_n \geq x - |x - x_n| > x - \frac{x}{2} = \frac{x}{2} > 0.$$

3.1.14. We show that $\lim(2n)^{1/n} = 1$. Clearly, $(2n)^{1/n} > 1$. Fix $\varepsilon > 0$, and show that $(2n)^{1/n} < 1 + \varepsilon$ for $n > N$ ($N \in \mathbb{N}$ depends on ε). Indeed, $(2n)^{1/n} < 1 + \varepsilon$ iff

$$2n < (1+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} \varepsilon^k.$$

Find $N \in \mathbb{N}$ s.t. $N > 4/\varepsilon^2 + 1$, or in other words, $2N < N(N-1)\varepsilon^2/2$. Then, for any $n > N$,

$$2n < \binom{n}{2} \varepsilon^2 < (1+\varepsilon)^n,$$

as desired.

3.1.16. For $n \geq 3$,

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots \cdot n \geq 1 \cdot 2 \cdot 3 \cdot 3 \ldots \cdot 3 = 2 \cdot 3^{n-2},$$

hence, for such n,

$$\frac{2^n}{n!} \leq 2\Big(\frac{2}{3}\Big)^{n-2} = \frac{2}{(1+1/2)^n} < \frac{2}{n \cdot 1/2} = \frac{4}{n}$$

(here, we use the techniques of Example (b), p. 58 of the textbook). For any $\varepsilon > 0$ find $N \in \mathbb{N}$ s.t. $N \geq 3$, and $1/N < \varepsilon/4$. Then $0 < 2^n/n! < \varepsilon$ for any $n > N$.

3.2.7. In Theorem 3.2.3, both sequences are assumed to converge, which need not happen in our case. Thus, that theorem is inapplicable here.

Find $A > 0$ s.t. $|b_n| < A$ for any n. Fix $\varepsilon > 0$, and show that there exists $N \in \mathbb{N}$ s.t. $|a_n b_n| < \varepsilon$ for any $n > N$. There exists $N \in \mathbb{N}$ s.t. $|a_n| < \varepsilon/M$ for $n > N$. For such n, $|a_n b_n| < \varepsilon$, and we are done.

3.2.9. Multiplying by the "conjugate," we get:

$$y_n = (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

hence $0 < y_n < 1/(2\sqrt{n})$. For any $\varepsilon > 0$, find $N \in \mathbb{N}$ s.t. $1/N < 4\varepsilon^2$. Then $|y_n| < \varepsilon$ for any $n > N$, hence $\lim_n y_n = 0$.

Let $y_n' = \sqrt{n} y_n$, and $x_n = 1/y_n' = 1 + \sqrt{1 + 1/n}$. By Theorem 3.2.10, $lim \sqrt{1 + 1/n} = 1$. Therefore, by Theorem 3.2.3, $\lim x_n = 2$, and furthermore, $\lim y_n' = 1/\lim x_n = 1/2$.

3.2.12. Let

$$x_n = \sqrt{(n+a)(n+b)} - n = (\sqrt{(n+a)(n+b)} - n) \cdot \frac{\sqrt{(n+a)(n+b)} + n}{\sqrt{(n+a)(n+b)} + n}$$
$$= \frac{n(a+b) + ab}{n\sqrt{(1+a/n)(1+b/n)} + n} = \frac{a + b + ab/n}{\sqrt{(1+a/n)(1+b/n)} + 1}.$$

But

$$\lim \sqrt{(1+a/n)(1+b/n)} = \sqrt{\lim(1+a/n)(1+b/n)} = 1,$$

hence $\lim \sqrt{(1+a/n)(1+b/n)} + 1 = 2$. Furthermore, $lim(a + b + ab/n) = a + b$, thus $\lim x_n = (a+b)/2$.

3.2.16. (a) Let $x_n = 1$ for each n. Then $x_{n+1}/x_n = 1$ for each n, and $\lim x_n = 1$.

(b) Let $x_n = n$. Then $x_{n+1}/x_n = 1$ for each n, and the sequence (x_n) is unbounded, hence divergent.

3.2.18. (d) $\lim(n!/n^n) = 0$. To establish this, let $k = \lfloor n/2 \rfloor$, and note that

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot \ldots \cdot k}{n^k} \cdot \frac{(k+1) \cdot \ldots \cdot n}{n^{n-k}} \leq \left(\frac{k}{n}\right)^k \leq 2^{-k} \leq 2^{1-n/2} = 2\big(2^{-1/2}\big)^n.$$

By Example 3.1.11(b), $\lim(2^{-1/2})^n = 0$, hence, by Squeeze Theorem, $\lim(n!/n^n) = 0$.

3.2.19. Fix $r \in (L, 1)$, and set $\varepsilon = r - L$. Then $|x_n^{1/n} - L| < \varepsilon$ for $n > N$, hence $x_n^{1/n} < L + \varepsilon = r$. Therefore, $0 \leq x_n \leq r^n$. An application of Squeeze Theorem and Example 3.1.11(b) shows that $\lim x_n = 0$.

3.3.2. We know that $x_1 > 1$, and $x_{n+1} = 2 - 1/x_n$. We use induction to show that this sequence is decreasing. More precisely, we show hat $1 < x_{n+1} < x_n$ for every n. To this end, it suffices to establish the following: if $a > 1$, then $1 < 2 - 1/a < a$. The left hand side inequality is obvious, and the right hand side is equivalent to $1 < (a + 1/a)/2$. However, $1 \leq (a + 1/a)/2$ by the Arithmetic-Geometric Mean Inequality. Moreover, the equality holds iff $a = 1/a$, which is not the case for us. Hence, $2 - 1/a < a$.

3.3.4. Assume for a moment that the sequence (x_n) has a limit, call it x. Then $x = \lim x_{n+1} = \lim \sqrt{2 + x_n} = \sqrt{2 + x}$, hence (solving the equation $x^2 = 2 + x$) $x = 2$.

We shall prove that, for each n, $x_n < x_{n+1} < 2$. To this end, it suffices to establish the following: if $0 < a < 2$, then $a < \sqrt{a+2} < 2$. However, $a + 2 > a + a = 2a > a \cdot a$, hence $\sqrt{a+2} > \sqrt{a^2} = a$. On the other hand, $\sqrt{2+a} < \sqrt{2+2} = 2$.

Thus, the sequence (x_n) is increasing, and bounded by 2 from above. As we have already seen, $\lim x_n = 2$.

3.3.11. Note that the sequence (y_n) is increasing. Indeed,

$$y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{(2n+1)(2n+2)} > 0.$$

On the other hand, y_n is a sum of n terms, each less than $1/n$, hence $y_n < 1$ for each n. Therefore, the sequence (y_n) is increasing, and bounded above. Therefore, it converges.

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