SOLUTIONS FOR HOMEWORK 5

3.3.13. (d) Let $x_n = (1 - 1/n)^n$, $y_n = (1 + 1/n)^n$, and $z_n = x_n y_n = (1 - 1/n^2)^n$. We know that $\lim y_n = e$. Show that $\lim z_n = 1$. Indeed, $z_n < 1$, and, by Bernoulli's Inequality (p. 29 of the textbook), $z_n \geq 1 + n \cdot (-1/n^2) = 1 - 1/n$. By Squeeze Theorem, $\lim z_n = 1$. Finally, $x_n = z_n / y_n$, hence $\lim x_n = \lim z_n / \lim y_n = 1/e$.

Remark. For any $t \in \mathbb{R}$, $e^t = \lim(1 + t/n)^n = \lim \sum_{k=0}^{n} t^k/k!$.

3.4.3. Let $x_n = f_{n+1}/f_n$, and $L = \lim x_n$. For $n \geq 2$,

$$x_n = \frac{f_{n+1}}{f_n} = \frac{f_n + f_{n-1}}{f_n} = \frac{f_{n-1}}{f_n} + 1 = 1 + \frac{1}{x_{n-1}}. \tag{1}$$

Note that the sequence (f_n) is increasing, hence $x_n \geq 1$, and, by (1), $x_n \leq 2$. Thus, $1 \leq L \leq 2$. Taking the limit of both sides of (1), we obtain:

$$L = \lim x_n = 1 + \lim \frac{1}{x_{n-1}} = 1 + \frac{1}{L}.$$

Solving the quadratic equation for L, we obtain $L = (1 + \sqrt{5})/2$.

3.4.10. Note that the sequence (s_n) is decreasing, hence it converges to $S = \inf s_n$. We have to find a sequence $n_1 < n_2 < \ldots$ s.t. $\lim_k x_{n_k} = S$. To this end, it suffices to establish the following: for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $n > N$ s.t. $|x_n - S| < \varepsilon$. Indeed, once the claim has been established, let $n_1 = 1$, and, for $k > 1$, find (inductively) $n_k > n_{k-1}$ s.t. $|x_{n_k} - S| < 2^{-k}$. Then $\lim_k x_{n_k} = S$.

So, suppose $\varepsilon > 0$ and $N \in \mathbb{N}$ are given. Then there exists $M > N$ s.t. $S \leq s_M < S + \varepsilon$. By definition, $s_M = \sup\{x_M, x_{M+1}, \ldots\}$, hence there exists $n \geq M$ s.t. $s_M - \varepsilon < x_n \leq S_M$. Therefore,

$$S - \varepsilon \leq s_M - \varepsilon < x_n \leq s_M < S + \varepsilon,$$

which implies $|x_n - S| < \varepsilon$.

3.4.11. Let $L = \lim(-1)^n x_n$, and show that $L = 0$. Indeed, let $y_n = (-1)^n x_n$. Then $\lim y_{2n} = \lim y_{2n-1} = L$. However, $y_{2n} \geq 0$ for any n, hence $L \geq 0$. On the other hand, $y_{2n-1} \leq 0$ for any n, which implies $L \leq 0$. Thus, $L = 0$.

Therefore, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|y_n| < \varepsilon$ for $n > N$. But $|y_n| = |x_n|$, hence $\lim x_n = 0$.

3.5.6. Let $x_n = \sum_{k=1}^{n} 1/k$. The, for any $p \in \mathbb{N}$,

$$x_{n+p} - x_n = \sum_{k=n+1}^{n+p} \frac{1}{k} < \frac{p}{n},$$

hence $\lim_n(x_{n+p} - x_n) = 0$ for any p. However, the sequence (x_n) is not Cauchy: for any $n \in \mathbb{N}$,

$$x_{2n} - x_n = \sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{n}{2n} = \frac{1}{2}.$$

3.5.11. For $n \geq 1$, let $x_n = y_{n+1} - y_n$. Then, for $n \geq 1$,

$$x_{n+1} = \Big(\frac{1}{3}y_n + \frac{2}{3}y_{n-1}\Big) - y_n = \frac{2}{3}(y_{n-1} - y_n) = -\frac{2}{3}x_n.$$

By Theorem 3.5.8, the sequence (y_n) converges. Furthermore, $x_n = (-2/3)^{n-1}x_1$ for any n, hence

$$\begin{aligned} y_n &= y_1 + \sum_{k=1}^{n-1} x_k = y_1 + x_1 \sum_{k=1}^{n-1} \Big(-\frac{2}{3}\Big)^{k-1} \\ &= y_1 + x_1 \frac{1 - (-2/3)^{n-1}}{1 - (-2/3)} = y_1 + \frac{3x_1}{5}\Big(1 - \Big(-\frac{2}{3}\Big)^{n-1}\Big). \end{aligned}$$

As $\lim(-2/3)^n = 0$, we conclude that

$$\lim y_n = y_1 + \frac{3}{5}(y_2 - y_1) = \frac{3y_2 + 2y_1}{5}.$$

3.5.13. $x_{n+1} = 2 + 1/x_n$, and $x_1 = 2$, hence $x_n \geq 2$ for any n. For $n \geq 2$

$$x_{n+1} - x_n = \frac{1}{x_n} - \frac{1}{x_{n-1}} = \frac{x_{n-1} - x_n}{x_n x_{n-1}}.$$

However, $x_n x_{n-1} \geq 4$, hence $|x_{n+1} - x_n| \leq |x_{n-1} - x_n|/4$, which implies that our sequence is contractive. Denote its limit (which exists by Theorem 3.5.8) by x. Then

$$x = \lim x_{n+1} = \lim \Big(2 + \frac{1}{x_n}\Big) = 2 + \frac{1}{x},$$

hence $x^2 - 2x - 1 = 0$. Solving this quadratic equation (and keeping in mind that $x \geq 2$), we conclude that $x = 1 + \sqrt{2}$.

3.7.8. Yes. If $\sum a_n$ converges, then $\lim a_n = 0$ (this follows from Cauchy's criterion for convergence). Let $b_n = a_n^2$. Then $\lim b_n/a_n = \lim a_n = 0$, hence $\sum b_n$ converges by Limit Comparison Test (Theorem 3.7.8).

3.7.9. NO. Let $a_n = 1/n^2$. Then $\sum a_n$ converges (Example 3.7.6(c)), while $\sum \sqrt{a_n}$ diverges (it is the harmonic series, described in Example 3.7.6(b)).

3.7.10. YES. Recall the Arithmetic-Geometric Mean Inequality: for non-negative x and y, $\sqrt{xy} \leq (x + y)/2$. Let $s_n = \sum_{k=1}^n a_k$, and $t_n = \sum_{k=1}^n \sqrt{a_k a_{k+1}}$. Note that the sequences (s_n) and (t_n) are increasing. Furthermore,

$$t_n \leq \sum_{k=1}^{n} \frac{a_k + a_{k+1}}{2} \leq s_{n+1} \leq \lim s_n,$$

hence it is bounded. As a bounded monotone sequence, (t_n) must converge.

3.7.11. Note that $b_n \geq x_n = a_1/n$, and $\sum x_n$ diverges (apply Limit Comparison Test to $\sum x_n$ and $\sum 1/n$). By Comparison test (3.7.7), $\sum b_n$ diverges.

Problem A (*a bonus problem – very little partial credit is given*): Prove that the Euler number e (defined in Example 3.3.6) is irrational.

For $n \in \mathbb{N}$, consider $a_n = \sum_{k=0}^{n} 1/k!$. We have proved in class that $a_1 < a_2 < \ldots < 3$, hence the sequence (a_n) converges. Moreover, $e = \lim a_n$.

Suppose, for the sake of contradiction, that e is rational. As $e \geq a_2 > 2$, we can write $e = M/N$, with $M, N \in \mathbb{N}$. Increasing N if necessary, we can assume $N \geq 2$ (here, N and M need not be mutually prime). Then $e = K/N!$, where $K = M(N-1)!$. Note that $a_N < e$, and we can write $a_N = K_N/N!$ for some positive integer N. Then $e - a_N = (K - K_N)/N! \geq 1/N!$.

Now let $b_n = a_n - a_N$. Then $\lim b_n = e - a_N \geq 1/N!$. However, for $n > N$,

$$b_n = \sum_{k=N+1}^{n} \frac{1}{k!} = \sum_{k=N+1}^{n} \frac{1}{1 \cdot \ldots N \cdot (N+1) \cdot \ldots \cdot k} \leq \frac{1}{N!} \sum_{k=N+1}^{n} \left(\frac{1}{N+1}\right)^{k-N},$$

hence, by the formula for the sum of the geometric sequence,

$$b_n \leq \frac{1}{N!} \frac{1/(N+1) - 1/(N+1)^{n-N+1}}{1 - 1/(N+1)} < \frac{1}{N \cdot N!}.$$

Therefore, $\lim b_n \leq 1/(N \cdot N!)$, which contradicts $\lim b_n = e - a_N \geq 1/N!$.

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