SOLUTIONS FOR HOMEWORK 8

5.4.2. (a) Note that, for $x, y \in \mathbb{R}\backslash\{0\}$,

$$f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2} = \frac{y^2 - x^2}{x^2y^2} = \frac{(y-x)(y+x)}{x^2y^2} = (y-x)\Big(\frac{1}{x^2y} + \frac{1}{xy^2}\Big).$$

If $x, y \in A$, then $x, y \geq 1$, hence $1/(x^2y) + 1/(xy^2) \leq 2$, and $|f(x) - f(y)| \leq 2|x - y|$. In particular, for a given $\varepsilon > 0$, $|f(x) - f(y)| < \varepsilon$ when $|x - y| < \varepsilon/2$. Thus, f is uniformly continuous on A.

(b) For $n \in \mathbb{N}$, let $x_n = 2^{-n}$ (clearly, $x_n \in B$). Then $x_n - x_{n+1} = 2^{-(n+1)}$. However, $f(x_{n+1}) - f(x_n) = 2^{2(n+1)} - 2^{2n} = 3 \cdot 4^n$.

Suppose, for the same of contradiction, that f is uniformly continuous on B. Then there exists $\delta > 0$ s.t. $|f(x) - f(y)| < 1$ whenever $x, y \in B$ satisfy $|x - y| < \delta$. Pick $n \in \mathbb{N}$ so large that $2^{-n} < \delta$. Then we must have $|f(x_n) - f(x_{n+1})| < 1$, which is false.

5.4.4. Note that, for $x, y \in \mathbb{R}$,

$$f(x) - f(y) = \frac{1}{1+x^2} - \frac{1}{1+y^2} = \frac{(y-x)(y+x)}{(1+x^2)(1+y^2)}.$$

Furthermore,

$$(1+x^2)(1+y^2) = 1 + x^2 + y^2 + x^2y^2 \geq \frac{1+x^2}{2} + \frac{1+y^2}{2} \geq |x| + |y|$$

(the last inequality follows from the Arithmetic-Geometric Mean Inequality). Therefore,

$$|f(x) - f(y)| \leq \frac{|y-x| \cdot |y+x|}{|x| + |y|} \leq |y - x|.$$

Thus, f is Lipschitz, hence uniformly continuous.

5.4.7. Both f and g are Lipschitz functions. Indeed, for f it is obvious. As far as g is concerned, recall that, for any t, $|\sin(t)| \leq |t|$. Furthermore,

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2},$$

hence

$$\big|\sin\alpha - \sin\beta\big| = 2\Big|\sin\frac{\alpha-\beta}{2}\Big| \cdot \Big|\cos\frac{\alpha+\beta}{2}\Big| \leq 2\frac{|\alpha-\beta|}{2} = |\alpha - \beta|.$$

Therefore, both f and g are uniformly continuous.

However, $h = fg$ is not uniformly continuous. Indeed, if it were, then there would exist $\delta > 0$ s.t. $|h(x) - h(y)| < 1$ whenever $|x - y| < \delta$. Pick $N \in \mathbb{N}$ s.t. $\pi/N < \delta$. Fix $M > N$. For $0 \leq k \leq N$, let $x_k = 2\pi M - \pi/2 + k\pi/N$. Then $x_0 = 2\pi M - \pi/2$, and

$x_N = 2\pi M + \pi/2$, hence $f(x_N) - f(x_0) = 4\pi M$. On the other hand, $|x_k - x_{k-1}| < \delta$ for $1 \le k \le N$, hence $|f(x_k) - f(x_{k-1})| < 1$, and, by the triangle inequality,

$$|f(x_N) - f(x_0)| \le \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| < N < M,$$

a contradiction.

5.4.8. Let $h = f \circ g$. We have to show that for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. $|h(x) - h(y)| < \varepsilon$ whenever $|x - y| < \delta$. As f is uniformly continuous, for any ε like this there exists $\sigma > 0$ s.t. $|f(s) - f(t)| < \varepsilon$ is $|s - t| < \sigma$. Moreover, g is uniformly continuous, hence there exists $\delta > 0$ s.t. $|g(x) - g(y)| < \sigma$ whenever $|x - y| < \delta$. Then $|f(g(x)) - f(g(y))| < \varepsilon$ if $|x - y| < \delta$, which is what we need.

5.4.9. Let $g = 1/f$. Then $g(x) - g(y) = (f(y) - f(x))/(f(x)f(y))$. For $\varepsilon > 0$, find $\delta > 0$ s.t. $|f(x) - f(y)| < k^2\varepsilon$ whenever $|x - y| < \delta$. For such x and y,

$$|g(x) - g(y)| \le \frac{|f(y) - f(x)|}{|f(x)| \cdot |f(y)|} < \frac{k^2\varepsilon}{k^2} = \varepsilon,$$

which establishes the uniform continuity of g.

5.4.13. *This is a bonus problem – very little partial credit is given.* Fixing $\varepsilon > 0$, we show that there exists $\delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ for any $x, y \in A$ s.t. $|x - y| < \delta$. We know that there exists a uniformly continuous function $g : A \to \mathbb{R}$ s.t. $|f(t) - g(t)| < \varepsilon/3$ for any $t \in S$. Find $\delta > 0$ s.t. $|g(x) - g(y)| < \varepsilon/3$ for any $x, y \in A$ s.t. $|x - y| < \delta$. By the triangle inequality, for such x and y we have:

$$|f(x) - f(y)| \le |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, the uniform continuity of f is established.

5.6.12. Suppose f is not strictly increasing on $[0, 1]$. Then there exist $0 \le a < b \le 1$ s.t. $f(a) \ge f(b)$. However, no value is attained twice, so $f(a) \ne f(b)$, hence $f(a) > f(b)$. Moreover, $f(0) < f(1)$, hence either $a > 0$, or $b < 1$. We henceforth assume that $a > 0$ (the other case is tackled similarly).

(1) $f(a) < f(0)$. Applying the Intermediate Value Theorem to $[0, a]$ and $[a, 1]$, we conclude that there exist $c_1 \in (0, a)$ and $c_2 \in (a, 1)$ s.t. $f(c_1) = f(c_2) = C$, where $C = (f(0) + f(a))/2$. Thus, f takes the value C twice, which is impossible.

(2) $f(a) = f(0)$. Impossible, too.

(3) $f(a) > f(0)$. Let $A = f(a)$, and $B = \max\{f(0), f(b)\}$. Note that $A > B$. Let $C = (A + B)/2$. Applying the Intermediate Value Theorem to $[0, a]$ and $[a, b]$, we conclude that there exist $c_1 \in (0, a)$ and $c_2 \in (a, b)$ s.t. $f(c_1) = f(c_2) = C$. Thus, f takes the value C twice, which is impossible.

5.6.13. Suppose, for the sake of contradiction, that f is continuous on $[0, 1]$. Then it attains its maximum and minimum, which we denote by M and m, respectively. Therefore, there exist $a_1 < a_2$ and $b_1 < b_2$ s.t. $f(a_1) = f(a_2) = M$, and $f(b_1) < f(b_2) < M$. Note that $m < M$ (otherwise, $f(x) = m = M$ for each $x \in [0, 1]$).

We show that some value is attained by f at least three times. To this end, consider the following three (mutually exclusive) cases:

(1) The intervals $[a_1, a_2]$ and $[b_1, b_2]$ partially overlap: $a_1 < b_1 < a_2 < b_2$, or $b_1 < a_1 < b_2 < a_2$.
(2) One of the intervals $[a_1, a_2]$ and $[b_1, b_2]$ contains the other: $a_1 < b_1 < b_2 < a_2$, or $b_1 < a_1 < a_2 < b_2$.
(3) The intervals $[a_1, a_2]$ and $[b_1, b_2]$ are disjoint: $a_1 < a_2 < b_1 < b_2$, or $b_1 < b_2 < a_1 < a_2$.

In case (1), suppose $a_1 < b_1 < a_2 < b_2$. Applying the Intermediate Value Theorem to the intervals $[a_1, b_1]$, $[b_1, a_2]$, and $[a_2, b_2]$, we show the existence of the points $c_1 \in [a_1, b_1]$, $c_2 \in [b_1, a_2]$, and $c_3 \in [a_2, b_2]$, s.t. $f(c_1) = f(c_2) = f(c_3) = (m + M)/2$. Thus, the value $(m + M)/2$ is attained at least three times, which is impossible.

In case (2), suppose $a_1 < b_1 < b_2 < a_2$. As f is continuous on $[b_1, b_2]$, there exists a point $c \in (b_1, b_2)$ where f attains its maximum on $[b_1, b_2]$. Note that $m < f(c) \leq M$. Applying the Intermediate Value Theorem to the intervals $[a_1, b_1]$, $[b_1, c]$, and $[c, b_2]$, we show the existence of the points $c_1 \in [a_1, b_1]$, $c_2 \in [b_1, c]$, and $c_3 \in [c, b_2]$, s.t. $f(c_1) = f(c_2) = f(c_3) = y$, where $y = (m + f(c))/2$. Indeed, $M = f(a_1) > y > f(b_1) = m$, hence the existence of c_1. The existence of c_2 and c_3 is established in the same way.

In case (3), suppose $a_1 < a_2 < b_1 < b_2$. By the continuity of f, there exists $c \in (b_1, b_2)$ where f attains its maximum on $[b_1, b_2]$. As in (2), $m < f(c) \leq M$. As before, an application of the Intermediate Value Theorem yields $c_1 \in [a_2, b_1]$, $c_2 \in [b_1, c]$, and $c_3 \in [c, b_2]$, s.t. $f(c_1) = f(c_2) = f(c_3) = y$, where $y = (m + f(c))/2$.

5.6.15. (a) We have to show that $x^r x^s = x^{r+s}$ when $r, s \in \mathbb{Q}$ (we already know this for $r, s \in \mathbb{Z}$). Write $r = a/b$ and $s = c/d$, with $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. We proved in class (see also Exercise 5.6.14) that x^r (or x^s) is well-defined – that is, the particular choice of a and b satisfying $a/b = r$ doesn't matter. Recalling that $t^m t^n = t^{m+n}$ for $m, n \in \mathbb{Z}$, and letting $t = x^{1/bd}$, $m = ad$, and $n = bc$, we have:

$$x^r x^s = x^{ad/bd} x^{bc/bd} = (x^{1/bd})^{ad} (x^{1/bd})^{bc} = (x^{1/bd})^{ad+bc} = x^{(ad+bc)/bd} = x^{r+s}.$$

(b) We have to show that $(x^r)^s = x^{rs}$ when $r, s \in \mathbb{Q}$ (this is known for $r, s \in \mathbb{Z}$). Note first that, for $m, n \in \mathbb{N}$ and $t \geq 0$,

$$\left(t^{1/m}\right)^{1/n} = t^{1/mn}. \qquad (1)$$

Indeed, $s = t^{1/mn}$ is the unique non-negative number such that $s^{mn} = t$. However,

$$\left(\left(t^{1/m}\right)^{1/n}\right)^{mn} = \left[\left(\left(t^{1/m}\right)^{1/n}\right)^n\right]^m = \left[t^{1/m}\right]^m = t,$$

which establishes (1).

The equality $(x^r)^s = x^{rs}$ is trivial if $r = 0$ or $s = 0$. Suppose first that both r and s are positive. As in (a), write $r = a/b$ and $s = c/d$, with $a, b, c, d \in \mathbb{N}$. Then

$$\left(x^r\right)^s = \left[\left(\left(x^{1/b}\right)^a\right)^{1/d}\right]^c.$$

Recall that $(t^m)^{1/n} = (t^{1/n})^m$ for $m, n \in \mathbb{N}$ (Theorem 5.6.7), hence, by (1),

$$\left(\left(x^{1/b}\right)^a\right)^{1/d} = \left(\left(x^{1/b}\right)^{1/d}\right)^a = \left(x^{1/bd}\right)^a.$$

Thus,

$$\left(x^r\right)^s = \left[\left(\left(x^{1/b}\right)^a\right)^{1/d}\right]^c = \left[\left(x^{1/bd}\right)^a\right]^c = \left(x^{1/bd}\right)^{ac} = x^{rs}.$$

Now suppose $r > 0$, $s < 0$. Then

$$\left(x^r\right)^s = \frac{1}{\left(x^r\right)^{-s}} = \frac{1}{x^{-rs}} = x^{rs}.$$

The case of $r < 0$ (and s either positive or negative) is tackled similarly.

Back to: the syllabus, the main page of the course, the assignment.