

## MATH 444 MIDTERM 1: SOLUTIONS FOR PRACTICE PROBLEMS

The test will be given on **Wednesday, October 8**. It will be based on Homeworks 1-5 (Chapters 1-3).

In preparing for the test, practice solving the problems from this list. In addition, take a look at the homework (at least one problem on the midterm will come directly from the homework), and at the examples given in the textbook.

**1.** The sequence  $(x_n)$  is defined as follows:  $x_1 = 1$ ,  $x_n = \sqrt{3 + 2x_{n-1}}$  for  $n \geq 2$ . Prove that this sequence converges, and find its limit.

We have to show that this sequence is contractive. For  $n \geq 2$ ,

$$\begin{aligned}x_{n+1} - x_n &= \sqrt{3 + 2x_n} - \sqrt{3 + 2x_{n-1}} \\ &= (\sqrt{3 + 2x_n} - \sqrt{3 + 2x_{n-1}}) \frac{\sqrt{3 + 2x_n} + \sqrt{3 + 2x_{n-1}}}{\sqrt{3 + 2x_n} + \sqrt{3 + 2x_{n-1}}} \\ &= \frac{2(x_n - x_{n-1})}{\sqrt{3 + 2x_n} + \sqrt{3 + 2x_{n-1}}}.\end{aligned}$$

Note that  $x_n \geq 0$  for each  $n$ , hence  $\sqrt{3 + 2x_n} + \sqrt{3 + 2x_{n-1}} \geq 2\sqrt{3}$ , hence  $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|/\sqrt{3}$ . Thus, the sequence is contractive, hence convergent. Denote its limit by  $x$ . Then  $x = \lim x_n = \lim \sqrt{3 + 2x_{n-1}} = \sqrt{3 + 2x}$ . Therefore,  $x^2 = 2x + 3$ . Solving this quadratic equation (and taking into account that  $x_n \geq \sqrt{3}$  for  $n \geq 2$ , hence  $x \geq \sqrt{3}$ ), we see that  $x = 3$ .

**2.** Prove that, for any non-negative integer  $n$ , 3 divides  $2 \cdot 4^n + 1$ .

For  $n \geq 0$ , let  $P(n)$  be the statement: 3 divides  $2 \cdot 4^n + 1$ . We use induction to prove that  $P(n)$  is true for every  $n$ .

The basic step is easy:  $2 \cdot 4^0 + 1 = 3$  is clearly a multiple of 3.

On the inductive step, we have to prove that, if  $P(n)$  is true, then  $P(n + 1)$  is true. That is, we have to prove that if 3 divides  $2 \cdot 4^n + 1$ , then 3 divides  $2 \cdot 4^{n+1} + 1$ . Note that  $4^{n+1} = 4 \cdot 4^n = 3 \cdot 4^n + 4^n$ , hence  $2 \cdot 4^{n+1} + 1 = 6 \cdot 4^n + (2 \cdot 4^n + 1)$ . By the induction hypothesis, the right hand side is divisible by 3.

**3.** Suppose  $a_1, a_2, \dots$  are positive numbers, and  $\sum a_k^2$  converges. Prove that  $\sum a_k/k$  converges.

*Hint.* Use the Arithmetic-Geometric Mean Inequality.

Fix  $\varepsilon > 0$ . We have to show that there exists  $N \in \mathbb{N}$  s.t.  $\sum_{k=n}^m a_k/k < \varepsilon$  for any  $m \geq n \geq N$ . By the Arithmetic-Geometric Mean Inequality,

$$\frac{a_k}{k} = \sqrt{a_k^2 \cdot 1/k^2} \leq \frac{a_k^2 + 1/k^2}{2}.$$

As the series  $\sum a_k^2$  and  $\sum 1/k^2$  converge (the convergence of the latter can be established by comparing it to  $\sum 1/(k(k+1))$ ), we can find  $N \in \mathbb{N}$  s.t.  $\sum_{k=n}^m a_k^2 < \varepsilon$  and  $\sum_{k=n}^m 1/k^2 < \varepsilon$  for any  $m \geq n \geq N$ . Clearly, this  $N$  works for our purposes.

4. Suppose  $a_1 > a_2 > \dots > 0$ , and  $\lim a_k = 0$ . Prove that  $\sum (-1)^{k+1} a_k$  converges.

*Hint.* As in Example 3.7.6(f), start by considering the even and odd partial sums separately.

Let  $s_n = \sum_{k=1}^n (-1)^k a_k$ . Then  $s_{2m+1} = s_{2m-1} - (a_{2m} - a_{2m+1}) < s_{2m-1}$ , and similarly,  $s_{2m+2} > s_{2m}$  for every  $m$ . Moreover,  $s_{2m-1} - s_{2m} = a_{2m} > 0$ . Therefore,

$$(1) \quad s_1 > s_3 > s_5 > \dots > s_6 > s_4 > s_2.$$

The sequences  $(s_{2m-1})$  and  $(s_{2m})$  are monotone and bounded, hence they converge to  $s$  and  $s'$ , respectively. Moreover,

$$s - s' = \lim(s_{2m-1} - s_{2m}) = \lim a_{2m} = 0,$$

which implies  $s = s'$ . It remains to show that  $\lim s_n = s$ . To this end, find an even  $N \in \mathbb{N}$  so that  $a_N < \varepsilon$ . By (1),  $s_N \leq s_n \leq s_{N+1} < s_N + \varepsilon$  for any  $n \geq n$ . Then  $s = \lim s_{sm} \in [s_N, s_N + \varepsilon]$ , hence  $|s_n - s| \leq \varepsilon$  for  $n \geq N$ .

5. Prove that, for any positive integer  $n$ ,

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{j=n+1}^{2n} \frac{1}{j}.$$

We prove this statement by induction on  $n$ . For  $n \in \mathbb{N}$ , let  $P(n)$  be the following statement:

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{j=n+1}^{2n} \frac{1}{j}.$$

The basic step consists of verifying  $P(1)$ . This is easy:  $1 - 1/2 = 1/2$ .

For the inductive step, we have to show that, if

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{j=n+1}^{2n} \frac{1}{j},$$

then

$$\sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} = \sum_{j=n+2}^{2n+2} \frac{1}{j}.$$

By the induction hypothesis,

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \frac{1}{n+1} + \sum_{j=n+2}^{2n} \frac{1}{j},$$

hence

$$\begin{aligned} \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} &= \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} + \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= \sum_{j=n+2}^{2n} \frac{1}{j} + \frac{1}{2n+1} + \frac{1}{n+1} - \frac{1}{2n+2} = \sum_{j=n+2}^{2n} \frac{1}{j} + \frac{1}{2n+1} + \frac{1}{2n+2}, \end{aligned}$$

as desired.

**6.** The sequence  $(x_n)$  is defined by the following rule:  $x_1 = x_2 = 1$ , and  $x_n = x_{n-1} + 2x_{n-2}$  for  $n \geq 3$ . Prove that, for any  $n \in \mathbb{N}$ ,  $x_n \geq 2^{n-2}$ .

Let  $P(n)$  be the statement that  $x_n \geq 2^{n-2}$ . We have to show that  $P(n)$  is true for any  $n$ . The proof proceeds by strong induction. The basic step consists of verifying  $P(n)$  for  $n = 1, 2$  (which is easy). For the inductive step, we establish the following: if  $P(k)$  is true for  $1 \leq k \leq n$  ( $n \geq 2$ ), then  $P(n+1)$  is also true. In other words, we have to show that  $x_{n+1} \geq 2^{n-1}$ , assuming that  $x_k \geq 2^{k-1}$  for any  $k \leq n$ . By the induction hypothesis,

$$x_{n+1} = x_n + 2x_{n-1} \geq 2^{n-2} + 2 \cdot 2^{n-3} = 2^{n-2} + 2^{n-2} = 2^{n-1}.$$

**7.** Consider the function  $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{N}$ , defined by  $f(a, b) = |a - b^2| + 1$ . Is  $f$  injective? Surjective?

$f$  is surjective: for any  $c \in \mathbb{N}$ ,  $f(c-1, 0) = c$ .  $f$  is not injective:  $f(0, 0) = f(1, 1)$ .

**8.** It is known that, for any  $x \in \mathbb{R}$ , there is a unique  $t \in \mathbb{R}$  satisfying  $t^3 = x$ . This  $t$  is denoted by  $x^{1/3}$ . Prove that, if a sequence  $(x_n)$  converges to  $x$ , then  $\lim x_n^{1/3} = x^{1/3}$ .

*Hint.* Recall that  $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ . Use  $a = x_n^{1/3}$  and  $b = x^{1/3}$  to show that, for  $n$  large enough,  $x_n^{1/3} - x^{1/3}$  must be “small.” It may be useful to consider the cases of  $x = 0$  and  $x \neq 0$  separately.

Note that

$$(2) \quad a^2 + ab + b^2 = \frac{(a+b)^2 + a^2 + b^2}{2}$$

hence  $a^2 + ab + b^2 \geq 0$ , with the equality attained iff  $a = b = 0$ . Therefore,  $a > b$  iff  $a^3 > b^3$ . Thus,  $x > y$  iff  $x^{1/3} > y^{1/3}$ .

Suppose first  $x = 0$ . Then, for any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  s.t.  $|x_n| < \varepsilon^3$  for  $n > N$ . Then  $|x_n^{1/3}| < \varepsilon$  for  $n > N$ . As  $\varepsilon > 0$  is arbitrary, we conclude  $\lim x_n^{1/3} = 0 = x$ .

Now suppose  $x \neq 0$ . By the above,

$$x_n^{1/3} - x^{1/3} = \frac{x_n - x}{x_n^{2/3} + x_n^{1/3}x^{1/3} + x^{2/3}}.$$

By (2),  $x_n^{2/3} + x_n^{1/3}x^{1/3} + x^{2/3} \geq x^{2/3}/2$ , hence

$$|x_n^{1/3} - x^{1/3}| \leq \frac{2}{x^{2/3}} |x_n - x|.$$

For any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  s.t.  $|x_n - x| < \varepsilon x^{2/3}/2$  for  $n > N$ . For such  $n$ ,  $|x_n^{1/3} - x^{1/3}| < \varepsilon$ .

**9.** Compute the following limits:

(a)  $\lim(n^2 + \cos n)/(2n^2 - 3) = 1/2$ . Indeed,

$$\lim \frac{n^2 + \cos n}{2n^2 - 3} = \lim \frac{n^2(1 + \cos n/n^2)}{n^2(2 - 3/n^2)} = \lim \frac{1 + \cos n/n^2}{2 - 3/n^2}.$$

However,  $\lim 1/n^2 = 0$ , hence  $\lim(2 - 3/n^2) = 2$ . Moreover,  $|\cos n/n^2| \leq 1/n^2$ , hence, by Squeeze Theorem,  $\lim \cos n/n^2 = 0$ , and  $\lim(1 + \cos n/n^2) = 1$ . Therefore,

$$\lim \frac{1 + \cos n/n^2}{2 - 3/n^2} = \frac{\lim(1 + \cos n/n^2)}{\lim(2 - 3/n^2)} = \frac{1}{2}.$$

(b) *Hint.* Recall that  $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ . Multiply and divide by the appropriate “conjugate.”

$\lim((n^3 + 3n^2)^{1/3} - n) = 1$ . Applying the Hint with  $a = (n^3 + 3n^2)^{1/3}$  and  $b = n$ , we get:

$$(3) \quad (n^3 + 3n^2)^{1/3} - n = \frac{3n^2}{(n^3 + 3n^2)^{2/3} + n(n^3 + 3n^2)^{1/3} + n^2}.$$

Furthermore,  $(n^3 + 3n^2)^{1/3} = n(1 + 3/n)^{1/3}$ , hence

$$\begin{aligned} (n^3 + 3n^2)^{1/3} - n &= \frac{3n^2}{n^2((1 + 3/n)^{2/3} + (1 + 3/n)^{2/3} + 1)} \\ &= \frac{3}{(1 + 3/n)^{2/3} + (1 + 3/n)^{2/3} + 1}. \end{aligned}$$

By Exercise 8,  $\lim(1 + 3/n)^{1/3} = 1$ . Therefore,

$$\lim((n^3 + 3n^2)^{1/3} - n) = \frac{3}{(\lim(1 + 3/n)^{1/3})^2 + \lim(1 + 3/n)^{1/3} + 1} = 1.$$

**10.** Suppose  $X$  and  $Y$  are sets,  $f : X \rightarrow Y$  is a map, and  $A, B$  are subsets of  $X$ . Prove that  $f(A) - f(B) \subseteq f(A - B)$ .

We have to show that any  $y \in f(A) - f(B)$  belongs to  $f(A - B)$ . Note that  $y$  belongs to  $f(A)$ , but not to  $f(B)$ . Therefore, there exists (not necessary unique)  $x \in A$  s.t.  $f(x) = y$ , and moreover,  $x \notin B$ . In other words,  $x \in A - B$ , hence  $y = f(x) \in f(A - B)$ .

**11.** Show that, for every  $n \in \mathbb{N}$ ,  $n! \leq ((n + 1)/2)^n$ .

*Hint.* You may use the Arithmetic-Geometric Mean Inequality.

Suppose first  $n$  is even, that is,  $n = 2k$  for some  $k$ . Then

$$(4) \quad n! = (1 \cdot n)(2 \cdot (n - 1)) \dots (k \cdot (k + 1)).$$

Thus,  $n!$  is a product of  $k$  terms of the form  $i(n+1-i)$  ( $1 \leq i \leq k = n/2$ ). By Arithmetic-Geometric Mean Inequality,

$$i(n+1-i) \leq \left(\frac{i+(n+1-i)}{2}\right)^2 = \left(\frac{n+1}{2}\right)^2.$$

As there are  $n/2$  terms like this in (4), we are done.

Now suppose  $n$  is odd, that is,  $n = 2k + 1$  for some  $k$ . Then

$$n! = (1 \cdot n)(2 \cdot (n-1)) \dots (k \cdot (k+2)) \cdot (k+1),$$

where  $k+1 = (n+1)/2$ . There are  $k = (n-1)/2$  terms of the form  $i(n+1-i)$  in the above product, each not exceeding  $(n+1)/2$ . Thus,

$$n! \leq \left(\frac{n+1}{2}\right)^{n-1} \frac{n+1}{2} = \left(\frac{n+1}{2}\right)^n.$$

**12.** Suppose a sequence  $(x_n)$  has the following property: if  $n_1 < n_2 < \dots$ , and  $n_k > 2^k$  for each  $k$ , then the sequence  $(x_{n_k})$  converges. Prove that the sequence  $(x_n)$  converges.

Consider a sequence  $(n_k)$  with  $n_k > 2^k$  for each  $k$ . Let  $x = \lim x_{n_k}$ . We shall show that  $x = \lim x_n$ . Indeed, otherwise, by Theorem 3.4.4, there exists  $\varepsilon > 0$  so that, for any  $N \in \mathbb{N}$ , there exists  $n > N$  with the property that  $|x_n - x| \geq \varepsilon$ . We construct the sequence  $(m_k)$  as follows: let  $m_1 = n_1$ . Suppose  $m_1, \dots, m_{2i-1}$  have already been selected in such a way that  $m_k > 2^k$  for  $1 \leq k \leq 2i-1$ . Find  $m_{2i} > \max\{2^{2i+1}, m_{2i-1}\}$  so that  $|x_{m_{2i}} - x| \geq \varepsilon$ . Then pick  $m_{2i+1}$  in such a way that  $m_{2i+1} = n_s$  for some  $s$ , and  $m_{2i+1} > m_{2i}$ . The sequence  $(x_{m_j})$  must converge. By construction,  $(x_{m_{2i-1}})$  is a subsequence of  $(x_{n_k})$ , hence  $\lim x_{m_{2i-1}} = x$ . Therefore,  $\lim x_{m_j} = x$ . On the other hand,  $|x_{m_{2i}} - x| \geq \varepsilon$ , which contradicts the above.

**13.** Prove that  $2^{1/3}$  is irrational.

Note that  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ , hence  $x^{1/3} > y^{1/3}$  iff  $x > y$ . In particular,  $2^{1/3} > 1^{1/3} = 1$ . If  $2^{1/3}$  is rational, we can write  $2^{1/3} = m/n$ , where  $m$  and  $n$  are mutually prime positive integers. Raising both sides to the power 3, and multiplying by  $n^3$ , we get:  $2n^3 = m^3$ . Thus, 2 divides  $m^3$ . As 2 is prime, it must divide  $m$ . Hence, we can write  $m = 2k$ , and  $n^3 = 4k^3$ . Thus,  $n$  is divisible by 2, which contradicts the assumption that  $m$  and  $n$  have no common factors.

**14.** Prove that, for any real number  $x$ , there exists a sequence  $(x_n)$  of rational numbers such that  $x_1 < x_2 < \dots$ , and  $\lim x_n = x$ .

By the density of  $\mathbb{Q}$ , for any  $n$  there exists  $x_n \in \mathbb{Q} \cap (x - 2^{-n}, x - 2^{-(n-1)})$ . In other words:

$$x - 2^{-1} < x_1 < x - 2^{-2} < x_2 < x - 2^{-3} < \dots$$

The sequence  $(x_n)$  has the desired properties.

**15.** Prove that there exists a family  $(A_x)_{x \in \mathbb{R}}$  of infinite subsets of  $\mathbb{N}$ , so that  $A_x \cap A_y$  is finite whenever  $x \neq y$ .

*Hint.*  $\mathbb{N}$  is equipollent with  $\mathbb{Q}$ , thus it suffices to find a family of sets  $B_x \subseteq \mathbb{Q}$  ( $x \in \mathbb{R}$ ) whose intersections are finite. These sets can be constructed using the sequences from Exercise 14.

By the previous exercise, for each  $x \in \mathbb{R}$  there exists a sequence of rational numbers  $(z_n^{(x)})_{n=1}^{\infty}$  s.t.  $z_1^{(x)} < z_2^{(x)} < \dots$ , and  $\lim z_n^{(x)} = x$ . Let  $B_x = \{z_1^{(x)}, z_2^{(x)}, \dots\}$  be the corresponding subset of  $\mathbb{Q}$ . Note that, for  $x \neq y$ ,  $B_x \cap B_y$  cannot be infinite. Indeed, there exists  $N \in \mathbb{N}$  s.t.  $|z_n^{(x)} - x| < |x - y|/2$  and  $|z_n^{(y)} - y| < |x - y|/2$  for  $n > N$ . Suppose  $z \in B_x \cap B_y$ . Then  $z = z_n^{(x)} = z_m^{(y)}$  for some  $m$  and  $n$ . By the triangle inequality, either  $m$  or  $n$  doesn't exceed  $N$ . Therefore,  $B_x \cap B_y \subseteq \{z_1^{(x)}, z_1^{(y)}, \dots, z_N^{(x)}, z_N^{(y)}\}$ , hence finite.

There exists a bijection  $f : \mathbb{Q} \rightarrow \mathbb{N}$ . The sets  $A_x = f(B_x)$  have the desired properties.

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To: the syllabus, the main page of the course, the problems.