MATH 444: SOLUTIONS FOR MIDTERM 2

1 *(10 points)*: Compute

$$\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1}$$

Note that, for $x \neq 1$,

$$\frac{\sqrt{x^2+3}-2}{x-1} = \frac{\sqrt{x^2+3}-2}{x-1} \cdot \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} = \frac{x^2-1}{(x-1)(\sqrt{x^2+3}+2)}$$
$$= \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+3}+2)} = \frac{x+1}{\sqrt{x^2+3}+2},$$

hence

$$\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1} = \lim_{x\to 1} \frac{x+1}{\sqrt{x^2+3}+2} = \frac{1+1}{\sqrt{1^2+3}+2} = \mathbf{\frac{1}{2}}.$$

2 *(10 points)*: For $x \in \mathbb{R}$, define $f(x) = x^3 + 4x + 1$.

(a) Prove that f is strictly increasing on $\mathbb{R}$.

$f'(x) = 3x^2 + 4 > 0$ for any $x \in \mathbb{R}$. Thus, f is strictly increasing.

(b) Consider $g = f^{-1}$ (the inverse function to f). Compute $g'(6)$.

Hint. $g(6)$ is an integer between -2 and 2.

By trial and error, we see that $f(1) = 6$, hence $g(6) = 1$. Then

$$g'(6) = \frac{1}{f'(g(6))} = \frac{1}{f'(1)} = \mathbf{\frac{1}{7}}.$$

3 *(10 points)*: For $x \in \mathbb{R}$, define $f(x) = x \cos x$. Prove that the function f is not uniformly continuous on $\mathbb{R}$.

Suppose, for the sake of contradiction, that f is uniformly continuous. Then there exists $\delta > 0$ s.t. $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Pick $N \in \mathbb{N}$ s.t. $\pi/N < \delta$. Fix $M > N$. For $0 \leq k \leq N$, let $x_k = 2\pi M + k\pi/N$. Then $x_0 = 2\pi M$, and $x_N = 2\pi M + \pi$, hence $f(x_N) - f(x_0) = (4M+1)\pi$. On the other hand, $|x_k - x_{k-1}| < \delta$ for $1 \leq k \leq N$, hence $|f(x_k) - f(x_{k-1})| < 1$, and, by the triangle inequality,

$$|f(x_N) - f(x_0)| \leq \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| < N < M,$$

a contradiction.

4 *(10 points)***:** Suppose a function f is differentiable on $\mathbb{R}$, and $f(n) = n(n-1)$ for any $n \in \mathbb{Z}$.

(a) Prove that, for any $n \in \mathbb{Z}$, there exists $x \in (n, n+1)$ with the property that $f'(x) = 2n$.

By Mean Value Theorem, there exists $x \in (n, n+1)$ s.t.

$$f'(x) = \frac{f(n+1) - f(n)}{(n+1) - n} = n(n+1) - n(n-1) = 2n.$$

(b) Prove that, for any $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ with the property that $f'(x) = y$.

Hint. You can use the result of part (a), even if you did not prove it.

If y is an even integer, we are done, by (a). Otherwise, find $n \in \mathbb{N}$ s.t. $2n < y < 2n+2$ (in fact, $n = \lfloor y/2 \rfloor$). By (a), there exist $x_1 \in (n, n+1)$ and $x_2 \in (n+1, n+2)$ s.t. $f'(x_1) = 2n$, and $f'(x_2) = 2n+2$. By Intermediate Value Theorem for Derivatives, there exists $x \in (x_1, x_2)$ s.t. $f'(x) = y$.

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