MATH 444 MIDTERM 2: PRACTICE PROBLEMS

The test will be given on **Wednesday, November 19**. It will be based on Homeworks 6-10, covering the material from Chapter 4 to Section 6.2 (Chapter 7 will not be included).

In preparing for the test, practice solving the problems from this list. In addition, take a look at the homework (at least one problem on the midterm will come directly from the homework), and at the examples given in the textbook.

1. Find

$$\lim_{x \to 0} \frac{\sqrt{1+4x^2}-1}{x^2+x^4}.$$

2. Suppose f is a uniformly continuous function on a finite open interval I. Prove that f is bounded on I, that is, there exists $M > 0$ s.t. $|f(x)| \leq M$ for any $x \in I$.

3. Suppose $0 < \alpha \leq 1$. A function $f : I \to \mathbb{R}$ (I is an interval) is called α-*Lipschitz* if there exists a constant K s.t. $|f(x) - f(y)| \leq K|x-y|^\alpha$ for any $x, y \in I$. 1-Lipschitz functions are called Lipschitz.

(a) Prove that any α-Lipschitz function is uniformly continuous.

(b) For $0 < \alpha < \beta \leq 1$, give an example of an α-Lipschitz which is not β-Lipschitz.

Hint. Consider $f(x) = x^\alpha$, defined on $[0, \infty)$.

(c – *the hardest part*) Give an example of a uniformly continuous function on $[0, 1]$, which is not α-Lipschitz for any $\alpha > 0$.

Hint. For $n = 0, 1, 2, \ldots$, let $x_n = 2^{-n}$, and $y_n = 2^{-n} + 2^{-n^2} = x_n + x_n^n$. Define $f : [0, 1] \to [0, 1]$ in such a way that $f(x) = x_{n-1}$ for $y_n \leq x \leq x_{n-1}$ ($n \in \mathbb{N}$), and $f(x) = a_n x + b_n$ for $x_n < x < y_n$. Select the sequences (a_n) and (b_n) to guarantee the continuity of f. Define $f(0)$ to obtain a continuous function on $[0, 1]$.

4. Prove that the function $f(x) = x^{1/5}$ ($x \in \mathbb{R}$) is not differentiable at 0.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|f(x)| < \varepsilon$ for any $x > N$. Prove that f is uniformly continuous.

6. Suppose the function f is continuous on the interval $[1, 9]$, differentiable in its interior, and satisfies $f(1) = 3$, $f(4) = 0$, and $f(9) = 10$. Prove that there exists $c \in (0, 1)$ s.t. $f'(c) = 1$.

7. Find the intervals where the function $f(x) = x^2(1-x)$ (defined on $[-1, 3]$) is increasing, and those on which it is decreasing.

8. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, and there exists a constant $\gamma \in (0, 1)$ s.t. $|f'(x)| \leq \gamma$ for any $x \in \mathbb{R}$. Consider a sequence $(s_n)_{n \in \mathbb{N}}$, defined recursively via $s_{n+1} = f(s_n)$, with s_1 given. Prove that the sequence (s_n) converges.

Hint. Recall Theorem 3.5.8.

9. Suppose f is defined on an interval I, and differentiable at the point a (belonging to the interior of I). Prove that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}.$$

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, and, for a certain $c \in \mathbb{R}$, $\lim_{x \to c} f'(x)$ exists. Prove that $f'(c) = \lim_{x \to c} f'(x)$.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Denote by f'' the *second derivative* of f: that is, $f'' = (f')'$. We say that f is *twice differentiable* at c if $f''(c)$ exists, that is, if f' is differentiable at c.

(a) Suppose $c \in \mathbb{R}$ is such that $f'(c) = 0$, f is twice differentiable at c, and $f''(c) > 0$. Prove that f has a relative minimum at c.

Hint. Consider the behavior of f' around c.

(b) Suppose f is twice differentiable on $\mathbb{R}$, and $f'' \geq 0$ everywhere. Prove that, for $x_1 < x_2 < x_3$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

(c) Suppose f is as in (b). Prove that, for any $a, b \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b).$$

Hint. Use the results of (b).

To: the syllabus, the main page of the course, the solutions.