MATH 444 MIDTERM 2: SOLUTIONS FOR PRACTICE PROBLEMS

The test will be given on **Wednesday, November 19**. It will be based on Homeworks 6-10, covering the material from Chapter 4 to Section 6.2 (Chapter 7 will not be included).

In preparing for the test, practice solving the problems from this list. In addition, take a look at the homework (at least one problem on the midterm will come directly from the homework), and at the examples given in the textbook.

1. Find

$$\lim_{x\to 0} \frac{\sqrt{1+4x^2}-1}{x^2+x^4}.$$

$$\frac{\sqrt{1+4x^2}-1}{x^2+x^4} = \frac{\sqrt{1+4x^2}-1}{x^2(1+x^2)} \frac{\sqrt{1+4x^2}+1}{\sqrt{1+4x^2}+1}$$
$$= \frac{4x^2}{x^2(1+x^2)(\sqrt{1+4x^2}+1)} = \frac{4}{(1+x^2)(\sqrt{1+4x^2}+1)},$$

hence

$$\lim_{x\to 0} \frac{\sqrt{1+4x^2}-1}{x^2+x^4} = \lim_{x\to 0} \frac{4}{(1+x^2)(\sqrt{1+4x^2}+1)} = \frac{4}{1\cdot(\sqrt{1}+1)} = \mathbf{2}.$$

2. Suppose f is a uniformly continuous function on a finite open interval I. Prove that f is bounded on I, that is, there exists $M > 0$ s.t. $|f(x)| \leq M$ for any $x \in I$.

We have $I = (a, b)$. We can extend f to a continuous function g on $[a, b]$. g is bounded, hence so is f.

3. Suppose $0 < \alpha \leq 1$. A function $f : I \to \mathbb{R}$ (I is an interval) is called *α-Lipschitz* if there exists a constant K s.t. $|f(x) - f(y)| \leq K|x - y|^\alpha$ for any $x, y \in I$. 1-Lipschitz functions are called Lipschitz.

(a) Prove that any α-Lipschitz function is uniformly continuous.

(b) For $0 < \alpha < \beta \leq 1$, give an example of an α-Lipschitz which is not β-Lipschitz.

Hint. Consider $f(x) = x^\alpha$, defined on $[0, \infty)$.

(c – *the hardest part*) Give an example of a uniformly continuous function on $[0, 1]$, which is not α-Lipschitz for any $\alpha > 0$.

Hint. For $n = 0, 1, 2, \ldots$, let $x_n = 2^{-n}$, and $y_n = 2^{-n} + 2^{-n^2} = x_n + x_n^n$. Define $f : [0, 1] \to [0, 1]$ in such a way that $f(x) = x_{n-1}$ for $y_n \leq x \leq x_{n-1}$ ($n \in \mathbb{N}$), and $f(x) = a_n x + b_n$ for $x_n < x < y_n$. Select the sequences (a_n) and (b_n) to guarantee the continuity of f. Define $f(0)$ to obtain a continuous function on $[0, 1]$.

(a) Suppose f is such that $|f(x) - f(y)| \le K|x-y|^\alpha$ for any $x, y \in I$. For $\varepsilon > 0$, let $\delta = (\varepsilon/K)^{1/\alpha}$. If $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

(b) Consider $f(x) = x^\alpha$ ($x \in [0, \infty)$).

(i) $|x^\alpha - y^\alpha| \le |x-y|^\alpha$, hence f is α-Lipschitz. To prove this, fix $c > 0$, and consider $f(t) = (t+c)^\alpha - t^\alpha$ ($t \ge 0$). f is continuous on its domain, and $f'(t) = \alpha((t+c)^{\alpha-1} - t^{\alpha-1}) < 0$ for any $t > 0$. Thus, f is decreasing, hence, for $t \ge 0$,

$$(t+c)^\alpha - t^\alpha = f(t) \le f(0) = c^\alpha.$$

Now suppose $0 \le x \le y$. Applying the above inequality with $t = x$ and $c = y - x$, we obtain $y^\alpha - x^\alpha \le (y-x)^\alpha$, which is what we need.

(ii) Suppose $\beta > \alpha$. Then f is not β-Lipschitz. Indeed, suppose it is. Then there exists a constant K s.t. $|x^\alpha - y^\alpha| \le K|x-y|^\beta$. In particular (letting $y = 0$), $x^\alpha \le Kx^\beta$ for any x, hence $x \le Cx^\gamma$, where $C = K^{1/\alpha}$, and $\gamma = \beta/\alpha > 1$. However, $\lim_{x\to 0} x^\gamma/x = \lim_{x\to 0} x^{\gamma-1} = 0$, which yields a contradiction.

(c) As suggested in the Hint, let $x_n = 2^{-n}$, and $y_n = 2^{-n} + 2^{-n^2} = x_n + x_n^n$. Define $f : [0,1] \to [0,1]$ in such a way that $f(x) = x_{n-1}$ for $y_n \le x \le x_{n-1}$, $f(0) = 0$, and $f(x) = x_n + 2^{n(n-1)}(x - x_n)$ for $x_n < x < y_n$. Clearly, f is continuous at 1, and on the intervals (x_n, y_n) and (y_n, x_{n-1}) for any n. Furthermore,

$$x_n + 2^{n(n-1)}(y_n - x_n) = 2^{-n} + 2^{n^2-n}2^{-n^2} == 2 \cdot 2^{-n} = 2^{-(n-1)} = x_{n-1},$$

hence

$$f(y_n) = x_{n-1} = 2^{-(n-1)} = \lim_{x\to y_n-} f = \lim_{x\to y_n+} f.$$

Therefore, f is continuous at y_n for every n. Similarly, one verifies the continuity at x_n. Finally, $f(x) \le x_n$ for $x \le x_n$, hence the continuity at 0. This shows that f continuous on $[0,1]$. As we are dealing with a closed finite interval, it is uniformly continuous.

Suppose, for the sake of contradiction, that f is α-Lipschitz, that is, there exists a constant K s.t. $|f(y) - f(x)| \le K|y-x|^\alpha$ for any $x, y \in [0,1]$. In particular,

$$2^{-n} = f(y_n) - f(x_n) \le K(y_n - x_n)^\alpha - 2^{-\alpha n^2},$$

which implies $n \ge \alpha n^2$ for any n. In other words, $1 \ge \alpha n$ for any $n \in \mathbb{N}$, which is impossible.

4. Prove that the function $f(x) = x^{1/5}$ ($x \in \mathbb{R}$) is not differentiable at 0.

By definition, f is differentiable at 0 iff there exists $\lim_{x\to 0} f(x)/x$. But $f(x)/x = 1/x^{4/5}$. For $n \in \mathbb{N}$, let $x_n = 1/n^5$. Then $\lim x_n = 0$, and $f(x_n)/x_n = 1/n^4$, hence the sequence $(f(x_n)/x_n)_{n\in\mathbb{N}}$ diverges.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|f(x)| < \varepsilon$ for any $x > N$. Prove that f is uniformly continuous.

We have to show that, for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. So, fix $\varepsilon > 0$, and find $N \in \mathbb{N}$ s.t. $|f(x)| < \varepsilon/2$ whenever

$|x| \geq N$. The function f is uniformly continuous on $[-(N+1), N+1]$, hence we can find $\delta \in (0,1)$ s.t. $|f(x) - f(y)| < \varepsilon$ if $x, y \in [-(N+1), N+1]$ satisfy $|x - y| < \delta$. Now suppose $x, y \in \mathbb{R}$ satisfy $|x-y| < \delta$. If both x and y belong to $[-(N+1), N+1]$, then $|f(x)-f(y)| < \varepsilon$. If one of these numbers (say x) lies outside of $[-(N+1), N+1]$, then $|x| > N+1$, and, by the triangle inequality, $|y| \geq |x| - \delta > N$. Therefore,

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

6. Suppose the function f is continuous on the interval $[1, 9]$, differentiable in its interior, and satisfies $f(1) = 3$, $f(4) = 0$, and $f(9) = 10$. Prove that there exists $c \in (0, 1)$ s.t. $f'(c) = 1$.

By Mean Value Theorem, there exists $c_1 \in (1, 4)$ s.t.

$$f'(c_1) = \frac{f(1) - f(4)}{1 - 4} = -1.$$

Similarly, there exists $c_2 \in (4, 9)$ s.t. $f'(c_2) = 2$. Applying Intermediate Value Theorem for derivatives to $[c_1, c_2]$, and observing that $-1 < 1 < 2$, we conclude the existence of $c \in (c_1, c_2)$ s.t. $f'(c) = 1$.

7. Find the intervals where the function $f(x) = x^2(1 - x)$ (defined on $[-1, 3]$) is increasing, and those on which it is decreasing.

The function f is continuous on $[-1, 3]$. Moreover,

$$f'(x) = 2x(1 - x) - x^2 = x(2(1 - x) - x) = 3x(2/3 - x).$$

Therefore, f' vanishes at 0 and $2/3$, is positive on $(0, 2/3)$, and negative on the intervals $(-1, 0)$ and $(2/3, 2)$. Therefore, f decreases on the intervals $[-1, 0]$ and $[2/3, 2]$, and increases on $[0, 2/3]$.

8. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, and there exists a constant $\gamma \in (0, 1)$ s.t. $|f'(x)| \leq \gamma$ for any $x \in \mathbb{R}$. Consider a sequence $(s_n)_{n \in \mathbb{N}}$, defined recursively via $s_{n+1} = f(s_n)$, with s_1 given. Prove that the sequence (s_n) converges.

Hint. Recall Theorem 3.5.8.

By Mean Value Theorem,

$$s_n - s_{n+1} = f(s_{n-1}) - f(s_n) = f'(c_n)(s_{n-1} - s_n),$$

with c_n lying between s_{n-1} and s_n. However, $|f'(c_n)| \leq \gamma < 1$, hence the sequence (s_n) is contractive: $|s_n - s_{n+1}| \leq \gamma |s_{n-1} - s_n|$ for any n. By Theorem 3.5.8, this sequences converges.

9. Suppose f is defined on an interval I, and differentiable at the point a (belonging to the interior of I). Prove that

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a - h)}{2h}.$$

We need to show that for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$\Big|\frac{f(a+h) - f(a-h)}{2h} - f'(a)\Big| < \varepsilon \text{ when } 0 < |h| < \delta.$$

We know that $\lim_{x\to a}(f(x) - f(a))(x-a) = f'(a)$, hence for any $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$\Big|\frac{f(x) - f(a)}{x-a} - f'(a)\Big| < \varepsilon \text{ when } 0 < |x-a| < \delta. \tag{1}$$

Observe that

$$2\frac{f(a+h) - f(a-h)}{2h} = \frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h},$$

and therefore,

$$\begin{aligned}&\frac{f(a+h) - f(a-h)}{2h} - f'(a)\\ &= \frac{1}{2}\Big(\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h} - 2f'(a)\Big)\\ &= \frac{1}{2}\Big(\Big(\frac{f(a+h) - f(a)}{h} - f'(a)\Big) + \Big(\frac{f(a) - f(a-h)}{h} - f'(a)\Big)\Big).\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}&\Big|\frac{f(a+h) - f(a-h)}{2h} - f'(a)\Big|\\ &\leq \frac{1}{2}\Big|\Big(\frac{f(a+h) - f(a)}{h} - f'(a)\Big| + \Big|\frac{f(a) - f(a-h)}{h} - f'(a)\Big|\Big).\end{aligned}$$

Combining the above inequality with (1) for $x = a + h$ and $x = a - h$, we conclude that

$$\Big|\frac{f(a+h) - f(a-h)}{2h} - f'(a)\Big| < \frac{1}{2}(\varepsilon + \varepsilon) = \varepsilon$$

if $0 < |h| < \delta$.

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, and, for a certain $c \in \mathbb{R}$, $\lim_{x\to c} f'(x)$ exists. Prove that $f'(c) = \lim_{x\to c} f'(x)$.

Let $a = \lim_{x\to c} f'(x)$. Suppose, for the sake of contradiction, that $b = f'(c) \neq a$. We can assume that $b > a$ (the other case is handled similarly). Let $\varepsilon = (b-a)/2$. Find $\delta > 0$ s.t. $|f'(x) - a| < \varepsilon$ whenever $0 < |x - c| \leq \delta$. Pick an arbitrary $d \in [c-\delta, c+\delta]\backslash\{c\}$, and let $a' = f(d)$. Note that

$$a' < a + \frac{b-a}{2} = \frac{b+2}{2} < b,$$

hence, by Intermediate value Theorem for Derivatives, there exists t, on (c, d) (or (d, c), depending on whether $c < d$ or $c > d$), s.t. $f'(t) = (a + b)/2$. In particular, $t \in [c-\delta, c+\delta]\backslash\{c\}$. But then, $f(t) < (b + a)/2$, which yields a contradiction.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Denote by f'' the *second derivative* of f: that is, $f'' = (f')'$. We say that f is *twice differentiable* at c if $f''(c)$ exists, that is, if f' is differentiable at c.

(a) Suppose $c \in \mathbb{R}$ is such that $f'(c) = 0$, f is twice differentiable at c, and $f''(c) > 0$. Prove that f has a relative minimum at c.

Hint. Consider the behavior of f' around c.

(b) Suppose f is twice differentiable on $\mathbb{R}$, and $f'' \geq 0$ everywhere. Prove that, for $x_1 < x_2 < x_3$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

(c) Suppose f is as in (b). Prove that, for any $a, b \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

Hint. Use the results of (b).

(a) Let $g = f'$. Then $g'(c) > 0$, hence $g = f'$ is increasing at c. That is, there exists $\delta > 0$ s.t. $f'(x) \geq f'(c) = 0$ for $x \in [c, c + \delta)$, and $f'(x) \leq f'(c) = 0$ for $x \in [c - \delta, c]$. By the first derivative criterion for relative extrema, f has a relative minimum at c.

(b) By Mean Value Theorem, there exists $c_1 \in (x_1, x_2)$ and $c_2 \in (x_2, x_3)$ s.t.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c_1) \ \text{ and } \ \frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(c_2). \tag{2}$$

However, $c_1 < c_2$. By Mean Value Theorem (applied to f'), there exists $c \in (c_1, c_2)$ s.t. $f'(c_2) - f'(c_1) = f'(c)(c_2 - c_1)$. But $f'(c) \geq 0$, hence $f'(c_2) \geq f'(c_1)$. Thus, (2) yields the desired result.

(c) Nov. 17: Error corrected. Suppose, without loss of generality, that $a < b$. The case of $\lambda \in \{0, 1\}$ is trivial, hence we can assume $0 < \lambda < 1$. Let $x_1 = a$, $x_3 = b$, and $x_2 = \lambda a + (1 - \lambda)b$. Then $x_1 < x_2 < x_3$, $x_2 - x_1 = (1 - \lambda)(b - a)$, and $x_3 - x_1 = \lambda(b - a)$. Applying (b), we obtain:

$$\frac{f(\lambda a + (1 - \lambda)b) - f(a)}{(1 - \lambda)(b - a)} \leq \frac{f(b) - f(\lambda a + (1 - \lambda)b)}{\lambda(b - a)}.$$

Therefore,

$$\lambda\big(f(\lambda a + (1 - \lambda)b) - f(a)\big) \leq (1 - \lambda)\big(f(b) - f(\lambda a + (1 - \lambda)b)\big).$$

We complete the proof by adding $\lambda f(a) + (1 - \lambda)f(\lambda a + (1 - \lambda)b)$ to both sides of the above equation.

To: the syllabus, the main page of the course, the problems.