Strichartz estimates for the Schrödinger and Wave equations on polygonal domains
Joint work with Matt Blair (UNM), G. Austin Ford (Stanford) and Sebastian Herr (U Bielefeld)

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Outline

Preliminaries

Quick Recap Strichartz Estimates for Schrödinger

Euclidean Surfaces with Conic Singularities

Wave Equations

Spectral Theory and Function Spaces
  Dispersive Estimates
  Reduction to dispersive estimates

Applications
  Morawetz Estimates on the Euclidean Cone
  Well-posedness for nonlinear waves
Let $B$ be a planar, polygonal domain, not necessarily convex. Let $V$ denote the set of vertices of $B$, and let $\Delta_B$ denote the Dirichlet or the Neumann Laplacian on $L^2(B)$. 
Applicable Billiards

Figure: Examples of polygonal billiards for which the Theorem is applicable with Dirichlet or Neumann boundary conditions on the solid lines and periodic boundary conditions on the dashed lines.
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A Review of Schrödinger Strichartz

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 u(0, x) = f(x).
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\]
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\[
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(D_t + \Delta) u(t, x) = 0 \\
u(0, x) = f(x).
\end{cases}
\]

Here, \( u \) satisfies either Dirichlet or Neumann homogeneous boundary conditions,

\[
u \bigg|_{[-T, T] \times \partial \Omega} = 0 \quad \text{or} \quad \partial_n u \bigg|_{[-T, T] \times \partial \Omega} = 0.
\]
The Strichartz Estimates

These are a family of space-time integrability bounds of the form

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C_T \|f\|_{H^s(\Omega)}$$

with $p > 2$ and $\frac{2}{p} + \frac{2}{q} = 1$. 
The Strichartz Estimates

More precisely, this self-adjoint operator possesses a sequence of eigenfunctions forming a basis for $L^2(\Omega)$. 

We write the eigenfunction and eigenvalue pairs as $\Delta \varphi_j = \lambda^2_j \varphi_j$, where $\lambda^2_j$ denotes the frequency of vibration.

The Sobolev space of order $s$ can then be defined as the image of $L^2(\Omega)$ under $(1 + \Delta)^{-s}$ with norm $\|f\|_2^{H_s(\Omega)} = \sum_{j=1}^{\infty} (1 + \lambda^{2j})^s |\langle f, \varphi_j \rangle|^2$.

Here, $\langle \cdot, \cdot \rangle$ denotes the $L^2$ inner product.
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The Sobolev space of order $s$ can then be defined as the image of $L^2(\Omega)$ under $(1 + \Delta)^{-s}$ with norm

$$\|f\|_{H^s(\Omega)}^2 = \sum_{j=1}^{\infty} \left(1 + \lambda_j^2\right)^s |\langle f, \varphi_j \rangle|^2.$$  

Here, $\langle \cdot, \cdot \rangle$ denotes the $L^2$ inner product.
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- In that case, one can take $s = 0$, and by scaling considerations, this is the optimal order for the Sobolev space; see for example Strichartz (1977), Ginibre and Velo (1985), Keel and Tao (1998), etc.
The Strichartz Estimates

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- In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates.
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- The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur.
- In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates.
- Nonetheless, estimates on general compact domains with smooth boundary have been shown by Anton (2008) and Blair-Smith-Sogge (2008). Both of these works build on the approach for compact manifolds of Burq-Gérard-Tzvetkov (2004).
Theorem

Let $\Omega$ be a compact polygonal domain in $\mathbb{R}^2$, and let $\Delta$ denote either the Dirichlet or Neumann Laplacian on $\Omega$. Then for any solution $u = \exp(-it\Delta)f$ to the Schrödinger IBVP with $f$ in $H^1_p(\Omega)$, the Strichartz estimates

$$\| u \|_{L^p([-T,T];L^q(\Omega))} \leq C_T \| f \|_{H^1_p(\Omega)}$$

hold provided $p > 2$, $q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$. 
Remarks

- In this work, the Neumann Laplacian is taken to be the Friedrichs extension of the Laplace operator acting on smooth functions which vanish in a neighborhood of the vertices.

- In this sense, our Neumann Laplacian imposes Dirichlet conditions at the vertices and Neumann conditions elsewhere.

- The Dirichlet Laplacian is taken to be the typical Friedrichs extension of the Laplace operator acting on smooth functions which are compactly supported in the interior of $\Omega$. 
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Remarks

- We note that our estimates have a loss of $s = \frac{1}{p}$ derivatives as in Burq-Gérard-Tzvetkov (2004), which we believe is an artifact of our methods.

- However, we also point out that in certain geometries a loss of derivatives is expected due to the existence of gliding rays, as shown by Ivanovici (2008).
Remarks

- We note that our estimates have a loss of $s = \frac{1}{p}$ derivatives as in Burq-Gérard-Tzvetkov (2004), which we believe is an artifact of our methods.

- Given specific geometries, there are results showing that such a loss is not sharp. For instance, when $\Omega$ is replaced by a flat rational torus, the Strichartz estimate with $p = q = 4$ holds for any $s > 0$, as was shown by Bourgain (1993); see also Bourgain (2007) for results in the case of irrational tori.
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- However, we also point out that in certain geometries a loss of derivatives is expected due to the existence of gliding rays, as shown by Ivanovici (2008).
Definition

A Euclidean surface with conical singularities (ESCS) is a topological space $X$ possessing a decomposition $X = X_0 \sqcup P$ for a finite set of singular points $P \subset X$ such that

1. $X_0$ is an open, smooth two-dimensional Riemannian manifold with a locally Euclidean metric $g$, and
2. each singular point $p_j$ of $P$ has a neighborhood $U_j$ such that $U_j \setminus \{p_j\}$ is isometric to a neighborhood of the tip of a flat Euclidean cone $C(S^1_{\rho_j})$ with $p_j$ mapped to the cone tip.
The Real Theorem

**Theorem**

Let $X$ be a compact ESCS, and let $\Delta_g$ be the Friedrichs extension of $\Delta_g \big|_{C_c^\infty(X_0)}$. Then for any solution $u = \exp(-it\Delta_g)f$ to the Schrödinger IVP on $X$ with initial data $f$ in $H^{\frac{1}{p}}(X)$, the Strichartz estimates

$$\|u\|_{L^p([-T,T];L^q(X))} \leq C_T \|f\|_{H^{\frac{1}{p}}(X)}$$

hold provided $p > 2$, $q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$. 
Proof Uses Square Function Estimates to Glue together the Flat and Conic Regions

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Proof Uses Square Function Estimates to Glue together the Flat and Conic Regions

- On the plane, estimates on the Schrödinger operator are well known.
- On the cone, we apply the Strichartz estimates for the Schrödinger operator on the Euclidean cone without loss from Ford (2009).
Proof Uses Square Function Estimates to Glue together the Flat and Conic Regions

- On the plane, estimates on the Schrödinger operator are well known.
- On the cone, we apply the Strichartz estimates for the Schrödinger operator on the Euclidean cone without loss from Ford (2009).
- Once we have the estimates on the propagator, the dyadic Strichartz estimate follows in a standard fashion.
Let $C(S^1_\rho)$ denote the flat cone over the circle of radius $\rho > 0$, defined as the product manifold $C(S^1_\rho) = \mathbb{R}_+ \times (\mathbb{R}/2\pi\rho\mathbb{Z})$ equipped with the (incomplete) metric $g(r, \theta) = dr^2 + r^2 \, d\theta^2$. In this work, we consider solutions $u : \mathbb{R} \times C(S^1_\rho) \to \mathbb{C}$ to the initial value problem for the wave equation on $C(S^1_\rho)$,

$$\begin{cases} 
(D_t^2 - \Delta_g) \, u(t, r, \theta) = G(t, r, \theta) \\
\quad u(0, r, \theta) = f(r, \theta) \\
\quad \partial_t u(0, r, \theta) = g(r, \theta). 
\end{cases}$$
Preliminaries for the Wave Equation

Let \( C(S^1_\rho) \) denote the flat cone over the circle of radius \( \rho > 0 \), defined as the product manifold \( C(S^1_\rho) = \mathbb{R}_+ \times (\mathbb{R}/2\pi \rho \mathbb{Z}) \) equipped with the (incomplete) metric \( g(r, \theta) = dr^2 + r^2 \, d\theta^2 \). In this work, we consider solutions \( u : \mathbb{R} \times C(S^1_\rho) \rightarrow \mathbb{C} \) to the initial value problem for the wave equation on \( C(S^1_\rho) \),

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\begin{cases}
(D^2_t - \Delta_g) u(t, r, \theta) = G(t, r, \theta) \\
u(0, r, \theta) = f(r, \theta) \\
\partial_t u(0, r, \theta) = g(r, \theta).
\end{cases}
\]

Here, we use \( \Delta_g \) to denote the Friedrichs extension of the nonnegative Laplace-Beltrami operator acting on \( C^\infty_c(C(S^1_\rho)) \), and we write \( D_t = \frac{1}{i} \partial_t \) for the Fourier-normalized time derivative.
Past Work

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- In the setting of exact cones, Cheeger and Taylor established the propagation of singularities and provided explicit formulae for solutions in terms of the functional calculus of the Laplace-Beltrami operator on the cross-section.
- This was then expanded upon by Melrose and Wunsch, who proved a propagation of singularities theorem for solutions to the wave equation in the more general setting of conic manifolds.
Preliminaries

The wave Strichartz estimates on the cone take the form

\[ \|u\|_{L^p(\mathbb{R}; L^q(C(S_1^1)))} + \|(u, \partial_t u)\|_{L^\infty(\mathbb{R}; H^\gamma(C(S_1^1)) \times H^{\gamma-1}(C(S_1^1)))} \leq \|(f, g)\|_{H^\gamma(C(S_1^1)) \times H^{\gamma-1}(C(S_1^1))} + \|G\|_{L^{p'}(\mathbb{R}; L^{q'}(C(S_1^1)))}. \]
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The exponents \(p, q, \tilde{p}, \tilde{q}, \gamma\) satisfy the scale invariant condition

\[ \frac{1}{p} + \frac{2}{q} = 1 - \gamma = \frac{1}{\tilde{p}'} + \frac{2}{\tilde{q}'} - 2 \]

as well as the admissibility requirements for both \((p, q)\) and \((\tilde{p}, \tilde{q})\)

\[ \frac{1}{p} + \frac{1}{2q} \leq \frac{1}{4} \quad \text{and} \quad 4 < p \leq \infty. \]
The wave Strichartz estimates on the cone take the form
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\| u \|_{L^p(\mathbb{R}; L^q(C(S_1^1)))} + \left\| (u, \partial_t u) \right\|_{L^\infty(\mathbb{R}; H^\gamma(C(S_1^1)) \times H^{\gamma-1}(C(S_1^1)))} \\
\lesssim \|(f, g)\|_{H^\gamma(C(S_1^1)) \times H^{\gamma-1}(C(S_1^1))} + \| G \|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(C(S_1^1)))}.
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\frac{1}{p} + \frac{1}{2q} \leq \frac{1}{4} \quad \text{and} \quad 4 < p \leq \infty.
\]

Note, these are the same as on Euclidean space! As for the Schrödinger equation, the Strichartz estimates are not altered by the diffraction point on the cone.
More History

- Strichartz inequalities are well-established for constant coefficient wave equations posed on $\mathbb{R}^n$ (see Strichartz, Ginibre-Velo, Lindblad-Sogge, Keel-Tao, and references contained therein).
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These were derived using frequency localized regularizations of the Schwartz kernel given by the fundamental solution to the wave equation. In particular, the results generally rely on oscillatory integration and Fourier restriction estimates.
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- These were derived using frequency localized regularizations of the Schwartz kernel given by the fundamental solution to the wave equation. In particular, the results generally rely on oscillatory integration and Fourier restriction estimates.

- However, only partial progress has been made in establishing these estimates for solutions on manifolds, domains, or singular spaces such as cones. In the last case, the conic singularity affects the flow of energy and complicates many of the known techniques for establishing these inequalities.
Explicit representations of the fundamental solution to the wave equation on $C(S^1)$ were developed by Cheeger and Taylor. Unlike the fundamental solution to the Schrödinger equation, however, the fundamental solution to the wave equation is unbounded, and as a consequence one must rework the $L^1 \to L^\infty$ dispersive estimates.
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Even on $\mathbb{R}^2$, there is no dispersive estimate analogous to the Schrödinger case which is valid for any choice of initial data, regardless of whether derivatives are incorporated to ensure scale invariance.
Explicit representations of the fundamental solution to the wave equation on $C(S^1_\rho)$ were developed by Cheeger and Taylor. Unlike the fundamental solution to the Schrödinger equation, however, the fundamental solution to the wave equation is unbounded, and as a consequence one must rework the $L^1 \rightarrow L^\infty$ dispersive estimates.

Even on $\mathbb{R}^2$, there is no dispersive estimate analogous to the Schrödinger case which is valid for any choice of initial data, regardless of whether derivatives are incorporated to ensure scale invariance.

One way to circumvent this problem is to prove $L^1 \rightarrow L^\infty$ estimates for frequency localized solutions, showing that whenever the initial data $(f, g)$ is spectrally localized to frequencies near $\mu > 0$ there is a replacement for the dispersive estimate.
Namely, Strichartz estimates may be proved if one shows the following:

\[
\beta \left( \mu - \frac{1}{\sqrt{\Delta g}} \right) f = f
\]
and

\[
\beta \left( \mu - \frac{1}{\sqrt{\Delta g}} \right) g = g
\]
for some smooth cutoff \( \beta \) with \( \text{supp}(\beta) \subset (\frac{1}{4}, 4) \).

Then

\[
\| U(t) g \|_{L^\infty(C(S^1 \rho))} \lesssim \mu \left( 1 + \mu |t| \right)^{-1/2} \| g \|_{L^1(C(S^1 \rho))}
\]

\[
\| \dot{U}(t) f \|_{L^\infty(C(S^1 \rho))} \lesssim \mu \left( 1 + \mu |t| \right)^{-1/2} \left( \mu \| f \|_{L^1(C(S^1 \rho))} + \| \nabla g f \|_{L^1(C(S^1 \rho))} \right)
\]
Preliminaries

- Namely, Strichartz estimates may be proved if one shows the following:

  - Suppose \( \beta(\mu^{-1}\sqrt{\Delta_g})f = f \) and \( \beta(\mu^{-1}\sqrt{\Delta_g})g = g \) for some smooth cutoff \( \beta \) with \( \text{supp}(\beta) \subset (\frac{1}{4}, 4) \).
Preliminaries

- Namely, Strichartz estimates may be proved if one shows the following:

- Suppose $\beta\left(\mu^{-1}\sqrt{\Delta_g}\right) f = f$ and $\beta\left(\mu^{-1}\sqrt{\Delta_g}\right) g = g$ for some smooth cutoff $\beta$ with $\text{supp}(\beta) \subset \left(\frac{1}{4}, 4\right)$.

- Then

\[
\|\mathcal{U}(t)g\|_{L^\infty(C(S_1^\rho))} \lesssim \mu \left(1 + \mu |t|\right)^{-1/2} \|g\|_{L^1(C(S_1^\rho))}
\]

\[
\left\|\mathcal{U}(t)f\right\|_{L^\infty(C(S_1^\rho))} \lesssim \mu \left(1 + \mu |t|\right)^{-1/2} \times \left(\mu \|f\|_{L^1(C(S_1^\rho))} + \|\nabla_g f\|_{L^1(C(S_1^\rho))}\right).
\]
We use the abbreviations

\[ U(t) \overset{\text{def}}{=} \sin \left( t \sqrt{\Delta_g} \right) \quad \text{and} \quad \dot{U}(t) \overset{\text{def}}{=} \cos \left( t \sqrt{\Delta_g} \right). \]
In this work, our main theorem shows that the frequency localized dispersive estimate holds and, by making use of the Hilbert transform, that this estimate is sufficient to yield the full range of Strichartz estimates.

**Theorem**

*Suppose the triples \((p, q)\) and \((\tilde{p}, \tilde{q})\) satisfy the admissibility condition for some \(0 \leq \gamma \leq 1\). Then any solution \(u\) to the wave equation will satisfy the Strichartz estimates provided the right hand side is finite.*
Preliminaries

- As a byproduct of the proof, we also show local estimates on solutions which instead involve inhomogeneous Sobolev spaces on the right-hand side. These estimates will play a role in the applications, where local estimates on planar domains are developed.
Corollary

Let $u$ be a solution to the inhomogeneous problem,

\[
\begin{cases}
\left(D_t^2 - \Delta_g\right) u(t, r, \theta) = G(t, r, \theta) \\
u(0, r, \theta) = f(r, \theta) \\
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\end{cases}
\]

Suppose the pairs $(p, q), (\tilde{p}, \tilde{q})$ satisfy the admissibility condition. Then for some implicit constant depending on $T$,

\[
\|u\|_{L^p([-T, T]; L^q(C(S^1_\rho)))} + \|(u, \partial_t u)\|_{L^\infty([-T, T]; H^\gamma(C(S^1_\rho)) \times H^{\gamma-1}(C(S^1_\rho)))} \\
\lesssim \|(f, g)\|_{H^\gamma(C(S^1_\rho)) \times H^{\gamma-1}(C(S^1_\rho))} + \|G\|_{L^{\tilde{p}'}([-T, T]; L^{\tilde{q}'}(C(S^1_\rho)))},
\]

whenever the right hand side is finite.
The expressions for the Schwartz kernels of $\mathcal{U}(t)$ derived by Cheeger and Taylor vary depending on into which of three regions of the spacetime $\mathbb{R} \times C(S^1_{\rho}) \times C(S^1_{\rho})$ their arguments fall; these regions are

- **Region I** $\overset{\text{def}}{=} \{ (t, r_1, \theta_1, r_2, \theta_2) : 0 < t < d_g((r_1, \theta_1), (r_2, \theta_2)) \}$,
- **Region II** $\overset{\text{def}}{=} \{ (t, r_1, \theta_1, r_2, \theta_2) : d_g((r_1, \theta_1), (r_2, \theta_2)) < t < r_1 + r_2 \}$,
- **Region III** $\overset{\text{def}}{=} \{ (t, r_1, \theta_1, r_2, \theta_2) : t > r_1 + r_2 \}$.
Here, the Riemannian distance function $d_g$ is

$$d_g((r_1, \theta_1), (r_2, \theta_2)) =
\begin{cases}
(r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2))^{1/2}, & |\theta_1 - \theta_2| \leq \pi \\
r_1 + r_2, & |\theta_1 - \theta_2| \geq \pi,
\end{cases}$$

provided the angular coordinates $\theta_1$ and $\theta_2$ are chosen so that $|\theta_1 - \theta_2|$ gives the distance between the two points on the circle.
Region I is the part of spacetime in which the propagator is identically zero owing to finite speed of the propagation of supports.
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- Region II is the regime in which waves propagate as they would on a smooth manifold, i.e. the region in which there has been no interaction between the main front and the cone tip.
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- Region I is the part of spacetime in which the propagator is identically zero owing to finite speed of the propagation of supports.
- Region II is the regime in which waves propagate as they would on a smooth manifold, i.e. the region in which there has been no interaction between the main front and the cone tip.
- Finally, Region III is the region in which there has been an interaction between the main front and the cone point.
The singularities, i.e. wavefront set, of the propagators in the transition between Regions I and II are entirely "geometric," to use the terminology of Melrose and Wunsch, which is to say that they propagate via geodesic flow.

Those in the transition between Regions II and III can be either geometric or "diffractive." Heuristically speaking, the geometric singularities in this transition are the limits of the geometric singularities in the transition between Regions I and II, and the diffractive singularities are those emerging radially from the cone point after a singularity has entered.
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Preliminaries

We now discuss the formulae for the Schwartz kernel of $\mathcal{U}(t)$ in the different regions of spacetime. As we remarked, in Region I

$$K^I_{\mathcal{U}(t)}(r_1, \theta_1; r_2, \theta_2) \equiv 0.$$
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$$K^I_{\mathcal{U}(t)}(r_1, \theta_1; r_2, \theta_2) \equiv 0.$$  

In Region II, it is given by

$$K^{II}_{\mathcal{U}(t)}(r_1, \theta_1; r_2, \theta_2) = \frac{1}{2\pi} \sum_j \left[ t^2 - r_1^2 - r_2^2 + 2r_1 r_2 \cos (\theta_1 - \theta_2 - j \cdot 2\pi \rho) \right]^{-\frac{1}{2}},$$

where the index $j$ ranges over integers such that

$$0 < |\theta_1 - \theta_2 - j \cdot 2\pi \rho| < \cos^{-1}\left( \frac{r_1^2 + r_2^2 - t^2}{2r_1 r_2} \right).$$
In Region III, it is given by

\[
K_{\mathcal{U}(t)}^{\text{III}}(r_1, \theta_1; r_2, \theta_2) = \frac{1}{2\pi} \sum_j \left[ t^2 - r_1^2 - r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2 - j \cdot 2\pi \rho) \right]^{-\frac{1}{2}}
\]

\[
- \frac{1}{4\pi^2 \rho} \int_0^{\cosh^{-1}\left(\frac{t^2-r_1^2-r_2^2}{2r_1 r_2}\right)} \left[ t^2 - r_1^2 - r_2^2 - 2r_1 r_2 \cosh(s) \right]^{-\frac{1}{2}}
\]

\[
\times \left\{ \frac{\sin\left(\frac{\pi+\theta_1-\theta_2}{\rho}\right)}{\cosh\left(\frac{s}{\rho}\right) - \cos\left(\frac{\theta_1-\theta_2+\pi}{\rho}\right)} + \frac{\sin\left(\frac{\pi-(\theta_1-\theta_2)}{\rho}\right)}{\cosh\left(\frac{s}{\rho}\right) - \cos\left(\frac{\theta_1-\theta_2-\pi}{\rho}\right)} \right\} \, ds,
\]

where \( j \) now ranges over

\[
0 < |\theta_1 - \theta_2 - j \cdot 2\pi \rho| < \pi.
\]
In Cheeger and Taylor’s work, they produce analogous formulae which are valid on any exact cone $C(Y)$. While the methods employed here will not apply to these cases—we rely on the explicit form of these expressions when $Y = S^1_\rho$, we do believe that the analogous range of Strichartz estimates will hold on a general exact cone.
In Cheeger and Taylor’s work, they produce analogous formulae which are valid on any exact cone $C(Y)$. While the methods employed here will not apply to these cases—we rely on the explicit form of these expressions when $Y = S^1_\rho$, we do believe that the analogous range of Strichartz estimates will hold on a general exact cone.

Moreover, we believe that these estimates should be accessible from Cheeger and Taylor’s formulae using estimates on the operators in the functional calculus of the Laplacian on the cross-section $Y$. 
We begin by briefly recalling Cheeger’s functional calculus for exact cones $C(Y)$. We will define the homogeneous Sobolev spaces $\dot{H}^s(C(Y))$ appearing in the Strichartz estimates via this calculus. We refer to Cheeger’s articles with Taylor or the second book of Taylor’s series for an in-depth discussion of the functional calculus as well as other applications.
Let $Y^n$ be a compact, boundaryless Riemannian manifold with metric $h$, and let $C(Y) \overset{\text{def}}{=} \mathbb{R}_+ \times Y$ be the half-cylinder over $Y$. We make $C(Y)$ an exact cone by equipping it with the incomplete Riemannian metric

$$g(r, y) = dr^2 + r^2 h(y).$$
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$$g(r, y) = dr^2 + r^2 h(y).$$

The nonnegative Laplacian on $C(Y)$ is thus

$$\Delta_g = -\partial_r^2 - \frac{n}{r} \partial_r + \frac{1}{r^2} \Delta_h,$$

where $\Delta_h$ is the nonnegative Laplacian on the cross-section $Y$. 

Spectral Theory
Writing \( \{ \mu_j \}_{j=0}^{\infty} \) for the eigenvalues of \( \Delta_h \) with multiplicity and \( \{ \varphi_j : Y \rightarrow \mathbb{C} \}_{j=0}^{\infty} \) for the corresponding orthonormal basis of eigenfunctions, we define the rescaled eigenvalues \( \nu_j \) by

\[
\nu_j \overset{\text{def}}{=} \left( \mu_j + \frac{(n-1)^2}{4} \right)^{\frac{1}{2}}.
\]

Note that \( \mu_0 = 0 \) and \( \nu_0 = \frac{n-1}{2} \) in our convention.
Henceforth, we take $\Delta_g$ to be the Friedrichs extension of the above Laplace-Beltrami operator acting on functions.
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As is well-known, suitable functions $G : \mathbb{R} \to \mathbb{C}$ give rise to operators $G(\Delta_g)$ via the spectral theorem.
By taking advantage of the product structure of the metric $g(r, y)$ and separation of variables, Cheeger showed that the Schwartz kernel of $G(\Delta_g)$, which we will write as $K_{G(\Delta_g)}$, has the form

$$K_{G(\Delta_g)}(r_1, y_1; r_2, y_2) = (r_1 r_2)^{-\frac{n-1}{2}} \sum_{j=0}^{\infty} \tilde{K}_{G(\Delta_g)}(r_1, r_2, \nu_j) \varphi_j(y_1) \overline{\varphi_j(y_2)},$$

where the radial coefficient $\tilde{K}_{G(\Delta_g)}(r_1, r_2, \nu_j)$ is given by

$$\tilde{K}_{G(\Delta_g)}(r_1, r_2, \nu) \overset{\text{def}}{=} \int_{0}^{\infty} G(\lambda^2) J_{\nu}(\lambda r_1) J_{\nu}(\lambda r_2) \lambda \, d\lambda.$$
Here, $J_\nu(z)$ is the Bessel function of order $\nu$,

$$J_\nu(z) \overset{\text{def}}{=} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(\nu + j + 1)} \left(\frac{z}{2}\right)^{\nu+2j}.$$
We can view these formulae as consequences of a “factoring” of the spectral measure on $C(Y)$ into tangential components and radial components. Indeed, the product $J_{\nu}(\lambda r)\varphi_j(y)$ is a solution to the Helmholtz equation

$$\left(\Delta_g - \lambda^2\right)(J_{\nu}(\lambda r)\varphi_j(y)) = 0.$$
Estimates

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$$(\Delta_g - \lambda^2)(J_{\nu_j}(\lambda r)\varphi_j(y)) = 0.$$ 

This naturally leads one to consider the Hankel transform of order $\nu_j$,

$$\mathcal{H}_{\nu_j}[b(r)](\lambda) \overset{\text{def}}{=} \int_0^\infty b(r) J_{\nu_j}(\lambda r) r \, dr,$$

a unitary map $L^2(\mathbb{R}_+, r \, dr) \longrightarrow L^2(\mathbb{R}_+, \lambda \, d\lambda)$ satisfying $\mathcal{H}_{\nu_j} \circ \mathcal{H}_{\nu_j} = \text{Id}$ (Taylor).
We use the functional calculus to define the Sobolev spaces appearing in the Strichartz Theorem and the subsequent corollaries. Since the precise structure of $\mathcal{H}$ is not used, the definitions here are meaningful in other contexts as well. Let $\{\beta_k\}_{k \in \mathbb{Z}}$ be a collection of smooth cutoffs satisfying for $\zeta \neq 0$,

$$
\sum_{k=-\infty}^{\infty} \beta_k(\zeta) \equiv 1, \quad \beta_k(\zeta) \overset{\text{def}}{=} \beta_0(2^{-k} \zeta)
$$

and

$$
\text{supp}(\beta_0) \subset \left( \frac{1}{\sqrt{2}}, 2\sqrt{2} \right).
$$
We also make use of the smooth cutoff to low frequency
\[ \tilde{\beta}_0(\zeta) \overset{\text{def}}{=} 1 - \sum_{k=1}^{\infty} \beta_k(\zeta). \]
For real $s$, we define the inhomogeneous Sobolev spaces as the completion of $C_c^\infty(C(Y))$ under the norm

$$
\|u\|_{H^s(C(Y))}^2 \overset{\text{def}}{=} \left\| \tilde{\beta}_0 \left( \sqrt{\Delta_g} \right) u \right\|_{L^2(C(Y))}^2 + \sum_{k=1}^{\infty} 2^{2ks} \left\| \beta_k \left( \sqrt{\Delta_g} \right) u \right\|_{L^2(C(Y))}^2
$$
Using the functional calculus, it is not hard to see that this is equivalent to taking the completion of $C^\infty_c(C(Y))$ with respect to the norm $\left\| (\text{Id} + \Delta_g)^{s/2} u \right\|_{L^2(C(Y))}$. The homogeneous Sobolev spaces of order $s$ are defined similarly, this time taking the completion of $\text{Dom}(\Delta_g^{s/2})$ under the norm

$$\|u\|_{\dot{H}^s(C(Y))}^2 \overset{\text{def}}{=} \sum_{k=-\infty}^{\infty} 2^{2ks} \left\| \beta_k \left( \sqrt{\Delta_g} \right) u \right\|_{L^2(C(Y))}^2.$$
For $u \in \text{Dom}\left(\Delta_{s,g}\right)$ this norm is equivalent to the one defined by $\left\| \Delta_{s,g}^1 u \right\|_{L^2(C(Y))}$, which implies the duality property $\dot{H}^{-s}(C(Y)) = \left(\dot{H}^s(C(Y))\right)'$. 
Proof of Theorem

In this section, we prove the Strichartz Estimates using an explicit formula for the kernel of the sine propagator $\mathcal{U}(t)$. As discussed in the introduction, we will take a Littlewood-Paley decomposition of the solution to establish frequency-localized dispersive estimates. These will allow for a regularization of $K_{\mathcal{U}(t)}$ to unit frequency that will overcome the complications coming from unboundedness of the fundamental solution.
Dispersion Estimates

- Given our definition of homogeneous Sobolev spaces, we may assume that without loss of generality, $u$ is a solution to the wave equation with initial data

$$(f, g) \in \text{Dom}\left(\Delta_g^{\frac{\gamma}{2}}\right) \times \text{Dom}\left(\Delta_g^{\frac{\gamma-1}{2}}\right).$$
Given our definition of homogeneous Sobolev spaces, we may assume that without loss of generality, \( u \) is a solution to the wave equation with initial data

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\]

Using the Littlewood-Paley frequency cutoffs \( \beta_{k}\left(\sqrt{\Delta_{g}}\right) \), we define

\[
u_{k}(t, \cdot) \overset{\text{def}}{=} \beta_{k}\left(\sqrt{\Delta_{g}}\right) u(t, \cdot).
\]
Dispersive Estimates

Since the frequency cutoffs commute with the Laplacian, the frequency localized solutions $\{u_k\}_{k \in \mathbb{Z}}$ satisfy the collection of IVPs

$$\begin{cases}
\left( D^2_t - \Delta_g \right) u_k(t, r, \theta) = G_k(t, r, \theta) \\
\quad u_k(0, r, \theta) = f_k(r, \theta) \\
\quad \partial_t u_k(0, r, \theta) = g_k(r, \theta),
\end{cases}$$

where $G_k(t, \cdot) \overset{\text{def}}{=} \beta_k \left( \sqrt{\Delta_g} \right) G(t, \cdot)$ and $(f_k, g_k) \overset{\text{def}}{=} \left( \beta_k \left( \sqrt{\Delta_g} \right) f, \beta_k \left( \sqrt{\Delta_g} \right) g \right)$. 

Dispersive Estimates

In order to reduce the Theorem to a frequency localized estimate, we use the following proposition.

Proposition
Let $1 < q < \infty$. For elements $a \in L^q(C(S^1_\rho))$, we have

$$\left\| \left( \sum_{k=-\infty}^{\infty} \left| \beta_k \left( \sqrt{\Delta_g} \right) a \right|^2 \right) \right\|_{L^2}^{1/2} \approx \| a \|_{L^q(C(S^1_\rho))},$$

with implicit constants depending only on $q$. 
The proof of this proposition is implicit in the arguments in the Schrödinger estimate; there, these squarefunction estimates as they are known are shown for functions on a Euclidean surface with conical singularities.
Dispersive Estimates

The estimate follows by establishing bounds on the supremum of the Schwartz kernel of $\beta\left(\sqrt{\Delta_g}\right)\mathcal{U}(t)$. On $\mathbb{R}^2$, this is typically accomplished by oscillatory integral methods. On flat cones, analogous oscillatory integrals appear to be very difficult to obtain.
Dispersive Estimates

- The estimate follows by establishing bounds on the supremum of the Schwartz kernel of $\beta(\sqrt{\Delta_g})\mathcal{U}(t)$. On $\mathbb{R}^2$, this is typically accomplished by oscillatory integral methods. On flat cones, analogous oscillatory integrals appear to be very difficult to obtain.

- Instead, we work entirely on the spatial domain, treating the Littlewood-Paley cutoffs as operators which regularize the corresponding kernels. The dispersive estimates will then follow by showing that the regularized kernel is essentially bounded by its average over a set of unit size.
Dispersive Estimates

- We may unify the formulae to obtain a kernel which is supported in the union of Regions II and III. It will be written as the sum of two terms which we (somewhat) informally describe as a “geometric” term and a “diffractive” term,

\[ K_{\mathcal{U}(t)}(t, r_1, \theta_1; r_2, \theta_2) = K_{\mathcal{U}(t)}^{\text{geom}}(t, r_1, \theta_1; r_2, \theta_2) + K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2). \]
Dispersive Estimates

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Some terms can be unified to form the geometric term

\[ K_{\mathcal{U}(t)}^{\text{geom}}(t, r_1, \theta_1; r_2, \theta_2) = \Psi(t, r_1, r_2, \theta_1 - \theta_2), \]

where \( \Psi \) is defined as

\[
\Psi(t, r_1, r_2, \theta) \overset{\text{def}}{=} \sum_{-\pi \leq \theta + j \cdot 2\pi \rho \leq \pi} \left[ t^2 - r_1^2 - r_2^2 + 2r_1r_2 \cos(\theta + j \cdot 2\pi \rho) \right]^{-\frac{1}{2}}. 
\]
Remark

The summation depends on the relative positions of the tangential (angular) variables in the Schwartz kernel. The total number of terms in the sum is no more than $1 + 1/\rho$. Also note that it vanishes for $|\theta| > \pi$. 
Dispersive Estimates

The diffractive term will be the remaining term; it is supported solely in Region III. To simplify our expression for it, we use the abbreviations

\[
\alpha \overset{\text{def}}{=} \frac{t^2 - r_1^2 - r_2^2}{2r_1 r_2} = \frac{t^2 - (r_1 + r_2)^2}{2r_1 r_2} + 1
\]

\[
\beta \overset{\text{def}}{=} \cosh^{-1}(\alpha)
\]

\[
\varphi_1 \overset{\text{def}}{=} \frac{\pi + (\theta_1 - \theta_2)}{\rho}
\]

\[
\varphi_2 \overset{\text{def}}{=} \frac{\pi - (\theta_1 - \theta_2)}{\rho}
\]
We now write $K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2)$ as

$$K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2) = -\frac{1_{(0,t)}(r_1 + r_2)}{4\pi^2 \rho (2r_1 r_2)^{1/2}} \times \int_0^\beta \left[ \alpha - \cosh(s) \right]^{-1/2} \left[ \frac{\sin(\varphi_1)}{\cosh(s/\rho) - \cos(\varphi_1)} + \frac{\sin(\varphi_2)}{\cosh(s/\rho) - \cos(\varphi_2)} \right] ds.$$
In this section, we present some applications of the above analysis. We begin by discussing Morawetz estimates on Euclidean cones and, more generally, metric cones. We then present Strichartz estimates for the appropriate wave equation IVPs on wedge domains, polygonal domains, and Euclidean surfaces with conic singularities. We close by discussing well-posedness results for nonlinear wave equations in these settings.
Following ideas in Burq-Planchon-Stalker-Tahvildar-Zadeh, we prove Morawetz, or local energy decay, estimates for the wave equation on Euclidean cones. To begin, we define the multiplier operator

\[(\Omega^s \phi)(r, \theta) = r^s \phi(r, \theta).\]
Recall from remark that our definition of the homogeneous Sobolev spaces $\dot{H}^s(C(S^1_\rho))$ is in terms of the spectral decomposition of the Laplace-Beltrami operator $\Delta_g$. We may thus define the following subspace,

$$
\dot{H}^s_{\geq m}(C(S^1_\rho)) \overset{\text{def}}{=} \left\{ f \in \dot{H}^s(C(S^1_\rho)) : \Pi_j f = 0 \text{ for } \nu_j < \nu_m \right\},
$$

where $\Pi_j$ is the spectral projector and $\nu_m$ is the $m$-th modified eigenvalue of $\Delta_g$. 

---

Morawetz
We have the following theorem.

**Theorem**

Let $m \geq 1$ be an integer and $0 < \alpha < \frac{1}{4} + \frac{1}{2} \nu_m$. Given a solution $u$ to the wave equation IVP, there exists a constant $C = C(m, \alpha)$ such that for all $f \in \dot{H}^\frac{1}{2} \geq m \left( C(S^1_\rho) \right)$ and $g \in \dot{H}^{-\frac{1}{2}} \geq m \left( C(S^1_\rho) \right)$, we have

$$\left\| \Omega^{-\frac{1}{2}-2\alpha} \Delta_g^{\frac{1}{4}-\alpha} u \right\|_{L^2(\mathbb{R} \times C(S^1_\rho))} \leq C \left( \left\| f \right\|_{\dot{H}^\frac{1}{2} \geq m \left( C(S^1_\rho) \right)} + \left\| g \right\|_{\dot{H}^{-\frac{1}{2}} \geq m \left( C(S^1_\rho) \right)} \right).$$
NLW

As an application of the Strichartz estimates above, we note that these inequalities can be used to show that a theorem of Lindblad and Sogge on $\mathbb{R}^2$ carries over to any of the contexts above (Euclidean cones, polygonal domains, ESCSs, and wedge domains). Specifically, we let $X$ denote any one of these manifolds, imposing Dirichlet or Neumann boundary conditions as appropriate. Consider the semilinear initial value problem

$$\begin{cases}
(D_t^2 - \Delta_g) v(t, x) = \pm |v|^{\kappa-1} v \\
v(0, x) = v_0(x) \in H^{\gamma}(X) \\
\partial_t v(0, x) = v_1(x) \in H^{\gamma-1}(X),
\end{cases}$$

where $\gamma = \gamma(\kappa) = \max \left( \frac{3}{4} - \frac{1}{\kappa-1}, 1 - \frac{2}{\kappa-1} \right)$, i.e.

$$\gamma = \gamma(\kappa) = \begin{cases}
\frac{3}{4} - \frac{1}{\kappa-1}, & 3 < \kappa \leq 5 \\
1 - \frac{2}{\kappa-1}, & 5 \leq \kappa < \infty.
\end{cases}$$
In general, the Sobolev index $\gamma(\kappa)$ is the lowest degree of regularity for which the semilinear problem is locally well-posed; see the ill-posedness results in Lindblad-Sogge and Christ-Colliander-Tao. The well-posedness problem is one of many important applications of Strichartz estimates. It is an example of how these inequalities efficiently handle the perturbative theory for these equations.
Theorem

Suppose $X$ is a 2-dimensional manifold where the local Strichartz estimates are valid. Then given any pair of initial data in $(v_0, v_1) \in H^\gamma(X) \times H^{\gamma-1}(X)$ there exists $T > 0$ and a unique solution of NLW satisfying

$$(v(t, \cdot), \partial_t v(t, \cdot)) \in C^0\left([0, T]; H^\gamma(X) \times H^{\gamma-1}(X)\right) \cap L^p\left([0, T]; L^{\frac{3}{2}\left(\kappa-1\right)}(X)\right),$$

where $p = \max\left(\frac{3}{\gamma(\kappa)}, \frac{3}{2}(\kappa - 1)\right)$. Furthermore, if $T^*$ denotes the maximal lifespan of the solution in $H^\gamma(X) \times H^{\gamma-1}(X)$, then either $T^* = \infty$ or

$$\|V\|_{L^{\frac{3}{2}\left(\kappa-1\right)}([0,T^*) \times X)} = \infty.$$
When global Strichartz estimates are available, we also have global existence for initial data is sufficiently small in $\dot{H}^\gamma(X) \times \dot{H}^{\gamma-1}(X)$. 
The local existence follows from a standard fixed point argument which makes use of the estimates

\[
\| u \|_{L^p([0,T];L^q(X))} + \| (u, \partial_t u) \|_{L^\infty([0,T];H^\gamma(X) \times H^{\gamma-1}(X))} \\
\lesssim \| (f, g) \|_{H^\gamma(X) \times H^{\gamma-1}(X)} + \| G \|_{L^{\frac{3}{2+\gamma}}([0,T];L^{\frac{6}{7-4\gamma}}(X))},
\]

where \( \gamma = \gamma(\kappa) \) as before and

\[
(p, q) = \begin{cases} 
\left( \frac{3}{\gamma}, \frac{6}{3-4\gamma} \right), & 3 < \kappa < 9 \\
\left( \frac{3\kappa}{2+\gamma}, \frac{6\kappa}{7-4\gamma} \right), & 9 \leq \kappa < \infty.
\end{cases}
\]
The remainder of the theorem now follows by the same considerations as in Theorems 5.1 and 5.2 in Lindblad-Sogge once one checks that $v \in L^{3/2(\kappa-1)}([0, T] \times X)$. This follows from the fact that the left hand side controls the $L^{3/2(\kappa-1)}([0, T] \times X)$ by Hölder’s inequality for the case $5 \leq \kappa < \infty$.