# 1 More examples from section 1.7

9 *For what values of $h$ is $\mathbf{v_3}$ in Span$\{\mathbf{v_1}, \mathbf{v_2}\}$ where*

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \mathbf{v_2} = \begin{pmatrix} -3 \\ 9 \\ -7 \end{pmatrix} \mathbf{v_3} = \begin{pmatrix} 5 \\ -7 \\ h \end{pmatrix}$$

As someone pointed out to me after class, my solution wasn't very clear. To compensate I will present you three different ways of solving this problem. The first is more straightforward, the second relies on a noticing some details, and the third is what I did in class.

**Solution one**

To say that $\mathbf{v_3}$ is in $Span\{\mathbf{v_1}, \mathbf{v_2}\}$ is to say that there exist constants $c_1, c_2$, at least one of them nonzero, such that

$$c_1\mathbf{v_1} + \mathbf{c_2}\mathbf{v_2} = \mathbf{v_3}$$

By remark in the blue box on page 34 this vector equation has the same solution set as the linear system whose augmented matrix is

$$\begin{pmatrix} 1 & -3 & 2 \\ -3 & 9 & -7 \\ 5 & -7 & h \end{pmatrix}$$

Multiplying the first column first by 3 and adding it to the second column then multiplying the first column by $-5$ and adding it to the third column we obtain the following matrix:

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 0 & -1 \\ 0 & -8 & -10+h \end{pmatrix}$$

Because this matrix is an augmented matrix of a linear system and the middle row is of the form [0 0 -1], Theorem 2 from section 1.2 tells us that this system is inconsistent. Thus, no solutions exist regardless of what values of $h$ we use.

**Solution 2**

Notice that $\mathbf{v_2} = \mathbf{-3v_1}$. This implies that in the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ there exists a $\mathbf{v_j}$, namely $\mathbf{v_2}$, which can be written in terms of the other $\mathbf{v_j}$'s with nonzero weights. Theorem 7 now implies that this set is linearly dependent. As the linear dependence that we wrote down does not involve $\mathbf{v_3}$, and therefore $h$, no matter what value of $h$ we pick, the set is always linearly dependent. Thus, there exist no value of $h$ for which $\mathbf{v_3}$ is in $Span\{\mathbf{v_1}, \mathbf{v_2}\}$.

**Solution 3**

Let's do a little thinking first. The next two paragraphs are concerned with some general situations:

Suppose that $\mathbf{v_3}$ is in $Span\{\mathbf{v_1}, \mathbf{v_2}\}$. Then we can write a linear dependence $c_1\mathbf{v_1} + \mathbf{c_2v_2} - \mathbf{v_3} = \mathbf{0}$ (this follows from the definition of *Span*). So under this assumption $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly dependent.

Now instead assume that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly dependent. It means that we have an equation of the form $c_1\mathbf{v_1} + \mathbf{c_2v_2} + \mathbf{c_3v_3} = \mathbf{0}$ with at least one of the $c_1, c_2, c_3$ nonzero. IF, and this is important, if $c_3$ is nonzero then we may rewrite this as

$$-c_3\mathbf{v_3} = c_1\mathbf{v_1} + \mathbf{c_2v_2}$$

implying that

$$\mathbf{v_3} = -\frac{c_1}{c_3}\mathbf{v_1} + -\frac{\mathbf{c_2}}{\mathbf{c_3}}\mathbf{v_2}$$

And thus $\mathbf{v_3}$ is in $Span\{\mathbf{v_1}, \mathbf{v_2}\}$. On the other hand, IF all solutions have $c_3 = 0$, then for the same reason we may never write $\mathbf{v_3}$ as a sum of $\mathbf{v_1}, \mathbf{v_2}$ with weights. Thus, $\mathbf{v_3}$ is NOT in $Span\{\mathbf{v_1}, \mathbf{v_2}\}$.

Back to our case. We have the matrix

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 9 & -7 \\ 5 & -7 & h \end{pmatrix}$$

and row reducing we get that $x_3 = 0$, and thus by the remarks above $\mathbf{v_3}$ is never in $Span\{\mathbf{v_1}, \mathbf{v_2}\}$. I apologize for the confusion at the end of class.

15 *Determine by inspection whether these vectors are linearly independent*

$$\mathbf{v_1} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \mathbf{v_2} = \begin{pmatrix} 2 \\ 8 \end{pmatrix} \mathbf{v_3} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \mathbf{v_4} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Theorem 8 on page 69 states that "If a set contains more vectors than there are entries in each vector, then the set is linearly independent. That is, any set $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$."

The proof is located on the same page and is easy to follow. Notice that in this problem we have 4 vectors, each having 2 entries. Thus, Theorem 8 implies that the set is linearly dependent.

26 *Describe the possible echelon forms of a* $4 \times 3$ *matrix* $A$, $A =$ *[*$\mathbf{a_1}$ $\mathbf{a_2}$ $\mathbf{a_3}$*] such that* $\{\mathbf{a_1}, \mathbf{a_2}\}$ *is linearly independent and* $\mathbf{a_3}$ *is not in* *Span*$\{\mathbf{a_1}, \mathbf{a_2}\}$.

I claim that in this case $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ are linearly independent. This follows from our assumptions as follows: note that since $\mathbf{a_3}$ is not in *Span*$\{\mathbf{a_1}, \mathbf{a_2}\}$, then I cannot write down an equation of the form

$$c_1\mathbf{a_1} + \mathbf{c_2}\mathbf{a_2} = \mathbf{a_3}$$

If $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ were linearly dependent then I would have

$$c_1\mathbf{a_1} + \mathbf{c_2}\mathbf{a_2} = -\mathbf{c_3}\mathbf{a_3}$$

with at least one of the $c_1, c_2, c_3$ nonzero. If $c_3$ was zero then the above equation would be

$$c_1\mathbf{a_1} + \mathbf{c_2}\mathbf{a_2} = \mathbf{0}$$

and since $\{\mathbf{a_1}, \mathbf{a_2}\}$ are linearly independent, by definition we must have $c_1 = c_2 = 0$. On the other hand, if $c_3$ was nonzero then we could divide both sides by $-c_3$ and get the equation

$$\frac{-c_1}{c_3}\mathbf{a_1} + \frac{-\mathbf{c_2}}{\mathbf{c_3}}\mathbf{a_2} = \mathbf{a_3}$$

Implying that $\mathbf{a_3}$ is in $Span\{\mathbf{a_1}, \mathbf{a_2}\}$. But we said that this cannot happen in our case! Thus, $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ are linearly independent.

We know that this happens if and only if given the matrix $A = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}]$, $Ax = 0$ has only the trivial solution, and this happens if and only if the matrix $A$ has a pivot in every column. Thus, any echelon form of $A$ must be

$$\begin{pmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{pmatrix}$$