

Math 225 M1, Additional homework, due 8/29/06

The goal of this exercise is to make a connection between solving systems of linear equations and geometry. We will stick to 2 and 3 dimensional spaces that you're all familiar with, but as we will see later in the course these ideas generalize nicely.

The story goes like this: Start with the xy -plane. A solution set of a linear equation (all the points in the plane that satisfy the equation) can be viewed as a subset of points in the plane. For example, the solution set of $x - y = 0$ is represented by the line $y = x$.

When you have more than one equation in your system, a solution set will have to satisfy *all* equations of the system. One way of getting at the solution set is then to consider the solution set of the first equation. From it, pick only the points that satisfy the second equation. From that, pick only the points that satisfy the third equation, etc.,. At the end you're left with a set of points that satisfies all the equations.

To visualize the process you can think of a system of two linear equations in the xy -plane. The solution set to the first one is some sort of a line. On that line, you pick the points that satisfy the second equation. What you're actually doing is intersecting the two lines. What could possibly happen? Well, the two lines could be parallel in which case there are no solutions (the system is inconsistent). They could intersect at one point (the system is consistent with a unique solution) or the two equations could determine the same line, in which case the intersection is the whole line (a consistent system with infinitely many solutions).

In general, a the set of solutions to a linear equation in n variables represents a $n - 1$ dimensional subset of an n -dimensional space. For example, a linear equation in the plane determines a line. A linear equation in 3 space determines a plane. Adding an equation to your system then causes the solution set to be intersected with some $n - 1$ dimensional hyperplane. In general, the intersection will be an $n - 2$ dimensional subset (two lines (1-dimensional) in a 2 dimensional plane usually intersect at a point, which is 0-dimensional.) So on average every time you add an equation to your system you cut down the dimension of the solution space by 1.

Think about what this says: if we're working in 3 dimensional space and have 3 equations in 3 variables x, y, z then we would normally expect a 0 dimensional solution set. This translates to: given 3 planes, what are their possible intersections? Usually two planes intersect at a line, and another plane will intersect that line at a point. Note, we're assuming that the planes

intersect at a line. It could be that the two planes intersect at a line and that the third plane is parallel to that line, in which case we have no solutions. I was only discussing the 'general' case.

The following exercise is supposed to help you understand what was just discussed.

- a To begin, suppose that we're working with the following system of linear equations in two variables x and y .

$$x - y = 3$$

Describe, as a subset (picture) in the $x - y$ plane, the solution set of this system.

Hint: This is simple. Rewrite the equation in a more familiar form...

- b Let's add another equation to our system. Consider the following system of linear equations in two variables x and y

$$x - y = 3 \tag{0.1}$$

$$-x - y = 3 \tag{0.2}$$

Write down the associated augmented matrix and solve it using elementary row operations. Give a geometric interpretation of the solution set.

Hint: Again, this is simple. Each equation determines some subset of the plane. The solution set of the system consists of those points that satisfy both equations. Geometrically, this is the ...

- c Now consider the following system of linear equations in **three** variables x , y , and z .

$$x - y = 3 \tag{0.3}$$

The set of solutions to this system of linear equations is a plane. Come up with your own understanding of that.

Hint: Each solution is of the form (s_1, s_2, s_3) . For example, $(6, 3, 0)$ is a solution, as is $(6, 3, 5)$.

d Describe (plane, line, point, etc.,) the solution set of each of the following 3 systems of equations in 3 variables x, y, z :

1

$$x + y - z = 5 \quad (0.4)$$

2

$$x = 5$$

$$z = 3$$

3

$$x = 5$$

$$z = 3$$

$$x + y = 7$$