

1 1.2

- 5 Describe the possible echelon forms of a nonzero 2×2 matrix. We approach the problem by noting that a 2×2 matrix can have either 1 or 2 pivot columns. If the matrix has no pivot columns then every entry in the matrix must be 0.

If our matrix has two pivot columns then it must have this form:

$$\begin{pmatrix} \blacksquare & * \\ 0 & \blacksquare \end{pmatrix}$$

If our matrix has one pivot column then we have a choice to make. If the first column is a pivot column then we must have this form:

$$\begin{pmatrix} \blacksquare & * \\ 0 & 0 \end{pmatrix}$$

Otherwise, if the pivot column is the second column, then our matrix has this form:

$$\begin{pmatrix} 0 & \blacksquare \\ 0 & 0 \end{pmatrix}$$

- 12 Find the general solutions of the system whose augmented matrix is

$$\begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{pmatrix}$$

First find the reduced row echelon form of the matrix. Notice that we already have a 1 in the first entry of the first row, so let's use that to clear the first column. To do this, add the first row to the third row to get

$$\begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{pmatrix}$$

Notice that x_2 , which corresponds to the second column, is a free variable. Now move onto the third column and clear everything below the 1 by multiplying the second row by 4 and adding it to the third row:

$$\begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that x_3 is a basic variable and x_4 is free. Further, this matrix is now in reduced echelon form. Since there are no rows of the form $[0 \ 0 \ 0 \ 0 \ b]$ with b nonzero and this is an augmented matrix associated to a system of linear equations, Theorem 2 implies that the system is consistent.

From this we can write down the general solution:

$$x_1 = 7x_2 - 6x_4 + 5$$

$$x_2 = \text{free}$$

$$x_3 = 2x_4 - 3$$

$$x_4 = \text{free}$$

- 15 a The system is consistent by Theorem 2. Theorem 2 also implies that since every variable is a basic variable, there exists a unique solution.
- b The system is inconsistent by theorem 2 because the last row corresponds to the equation $0 = b$ where b is nonzero.
- 17 *Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system*

$$\begin{pmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{pmatrix}$$

By Theorem 2 we know that a system is consistent if and only if an echelon form of the augmented matrix has no rows of the form $[0 \ 0 \ b]$ for a nonzero b .

Begin by reducing the matrix to any echelon form. I chose to multiply the first row by -2 and add it to the second row:

$$\begin{pmatrix} 2 & 3 & h \\ 0 & 0 & 7 - 2h \end{pmatrix}$$

Hence by theorem 2 we see that the system is consistent if and only if $7 - 2h = 0$, which means that $h = \frac{7}{2}$.

21 *Determine if the statement is true or false. Justify your answer*

a *In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations*

False. The **reduced** echelon form is unique. You may cite the text for a justification, but there is a proof in the appendix in case you're curious.

b *The row reduction algorithm applies only to augmented matrices for a linear system*

False. As I've mentioned in class, we may use matrices to represent systems of linear equations, but row reduction is something that can be performed on any matrix.

c *A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix*

True. This is the definition.

d *Finding a parametric description of the solution set of a linear system is the same as solving the system*

True. For example, suppose that we have the following parametric description of a the solution set:

$$\begin{aligned} x_1 &= 2x_3 - 7 \\ x_2 &= 5 \\ x_3 &= \text{free} \end{aligned}$$

Then for each value of x_3 that we chose, we may write down the 3-tuple $(2x_3 - 7, 5, x_3)$ which satisfies the system. Also, every 3-tuple that satisfies the system will be of this form.

e *If one row in an echelon form of an augmented matrix is $[0 \ 0 \ 0 \ 5 \ 0]$ then the associated linear system is inconsistent*

False. Suppose that this is the augmented matrix of some linear system. Then the linear system is $x_4 = 0$ in x_1, \dots, x_4 , which clearly has a solution $(0, 0, 0, 0)$ and hence is consistent.

23 *Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not*

Since there are 3 pivot columns and the matrix has 3 rows, it must be that there's a pivot in each row. Hence the augmented matrix does not contain any rows of the form $[0 \ 0 \ 0 \ 0 \ 0 \ b]$ for a nonzero b . Theorem 2 now implies that the system is consistent. Note how I transitioned from the coefficient matrix to the augmented matrix: The coefficient matrix is the same as the augmented matrix except that the augmented matrix has an extra column at the end. Thus, a pivot in the last column of the coefficient matrix corresponds to a pivot in the next to last column of the augmented matrix.

24 *Suppose a system of linear equations has a 3×5 augmented matrix whose fifth column is a pivot column. Is the system consistent?*

No. The system is inconsistent because having a pivot in the last column of the augmented matrix means that any echelon form of the augmented matrix will have a last row of the form $[0 \ 0 \ 0 \ 0 \ b]$ for some nonzero b . Theorem 2 now implies that the system is inconsistent.

26 *Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.*

If the coefficient matrix has a pivot in every row, then none of the rows of the coefficient matrix will contain all zeroes. Hence, none of the rows of the augmented matrix will contain rows of the form $[0 \ 0 \ \dots \ 0 \ b]$. Theorem 2 implies that the system is consistent.

27 *Restate the last sentence in Theorem 2 using the concept of pivot columns: "If a linear system is consistent, then the solution is unique if and only if..."*

...every column of the coefficient matrix is a pivot column.

28 *What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?*

We would have to know that except for the last column of the augmented matrix, every column has to be a pivot column, **and** the last column is not a pivot column. This follows from Theorem 2, which asserts that if there are no free variables, then there exists a unique solution.