Math 220 BE1 Practice Exam 3

No calculators, books, cell phones, or notes are to be used during the test. This exam covers sections 4.2 through 4.9. You must show your work to recieve credit. Answer **all** questions.

1) State in full

(a) The Mean Value Theorem

Suppose that f is continuous on the closed, bounded interval $[a,b]$ and differentiable on the open interval (a,b). Then there is a number c between a and b for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) The Extreme Value Theorem

Let f be continuous on the closed, bounded interval $[a,b]$. Then f assumes both a maximum value and a minimum value somewhere on $[a,b]$.

(c) Rolle's Theorem

Suppose that f is continuous on $[a,b]$ and differentiable on (a,b) and that $f(a) = f(b)$. Then, for some c between a and b, $f'(c) = 0$.

(d) The Intermediate Value Theorem

Let f be continuous on the closed, bounded interval $[a,b]$, and let y be any number between $f(a)$ and $f(b)$. Then, for some input c between a and b, $f(c) = y$.

2) Find the following limits

(a) $\lim_{x\to\infty} e^{-x}\ln(x)$

$$\begin{aligned}\lim_{x\to\infty} e^{-x}\ln(x) &= \lim_{x\to\infty} \frac{\ln(x)}{\frac{1}{e^x}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} \\ &= \lim_{x\to\infty} \frac{1}{xe^x} = 0\end{aligned}$$

(b) $\lim_{x\to 0} \frac{1}{\cos(x)}$

$$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\cos(0)} = 1$$

(c) $\lim_{x\to\pi} \frac{\cos(x-\pi)-1}{(x-\pi)^3}$.

$$\begin{aligned} \lim_{x\to\pi} \frac{\cos(x-\pi)-1}{x-\pi)^3} &= \lim_{x\to\pi} \frac{-\sin(x-\pi)}{3(x-\pi)^2} \\ &= \lim_{x\to\pi} \frac{-\cos(x-\pi)}{6(x-\pi)} \end{aligned}$$

We cannot apply L'Hopital's rule since the numerator goes to 1. From the left-hand side, the limit is ∞, and from the right hand side the limit is $-\infty$ thus the limit does not exist.

3) (a) A triangle has legs on the positive x-axis and y-axes, and its hypotenuse passes through the point $(2,1)$. Which such triangle has the smallest area?

We first find the equation of a line passing through $(2,1)$ as a function of the slope m: $y-1=m(x-2)$ or equivalently $y=mx-2m+1$. Notice that $m<0$ if we want the triangle to have its legs on the positive x and y-axies.

Given such a line, the height of the triangle is $-2m+1$, and the length of the base is $(2m-1)/m$. Using the formula for the area of a right triangle we see that $A(m) = \frac{(2m-1)(1-2m)}{2m} = -2m+2-\frac{1}{2m}$. If we set $\frac{dA}{dm}=0$, we see that $-2+.5m^{-2}=0$, which implies that $m=-0.5$. The first derivative test indicates that this is a minimum, and it implies that the base is 4 and the height is 2.

*(b) Find all the critical points of $|x^2-x|$. *Hint: Rewrite the function as a piece-wise function to remove the absolute value.*

We first break this up into a piece-wise function:

$$|x^2-x| = \begin{cases} x^2-x & x<0 \\ -x^2+x & 0\le x\le 1 \\ x^2-x & x>1 \end{cases}$$

This means that the derivative is given by:

$$(|x^2-x|)' = \begin{cases} 2x-1 & x<0 \\ -2x+1 & 0<x<1 \\ 2x-1 & x>1 \end{cases}$$

Thus the derivative is 0 when $x=.5$, and undefined when $x=0$ and $x=1$. These are the critical values.

4) Graph the parametric curve given by $x=2\cos(t)$, $y=t$, $-\pi\le t\le\pi$. Label both axes and indicate the points corresponding to $t=0$ and $t=-\pi$ on the graph.

5) A lamp positioned on top of a 12 foot pole casts a shadow on a 6 foot man who is walking away from the lamp at the rate of 1 foot per second. How fast is the length of the shadow changing when the man is 8 feet away from the lamp?

Drawing the usual triangle picture, letting $x(t)$ denote the man's distance from the lamp and $s(t)$ the length of the shadow, we see that $\frac{12}{x+s} = 6s$. Thus $x = s$, and $x' = s'$. So at all times the shadow is changing at 1 foot per second.

6) (a) Find a third-order Taylor polynomial for $f(x) = e^x$ expanded around $x_0 = 0$.

The derivatives of e^x are all e^x, and $e^0 = 1$, so plugging that into our Taylor Polynomial formula we get that

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

(b) Use the polynomial from part a to estimate the value of e.

Since $e = e^1$, we look at $e^1 \approx T_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2 + \frac{2}{3}$.

7) Let $f(x) = |x - 1|$.

(a) Does f satisfy the hypothesis of the Intermediate Value Theorem on the interval $[-1, 1]$? Justify your answer.

Yes, f is continuous on the closed and bounded interval $[-1, 1]$.

(b) Does f satisfy the hypothesis of the Extreme Value Theorem on the interval $[-1, 1]$? Justify your answer.

Yes, f is continuous on the closed and bounded interval $[-1, 1]$.

(c) Does f satisfy the *conclusion* of the Mean Value Theorem on the interval $[-1, 1]$? Justify your answer.

No: $\frac{f(1)-f(-1)}{1-(-1)} = 2/2 = 1$, and for no value of c on the interval $(-1, 1)$ is $f'(c) = 1$.