Math 220. Lab 5. Fall 2007

Mastery Exam Practice

Lab 5 Problems

(1) Compute the derivative of $f(x) = x^2 - 5$ using the definition of the derivative.

Here's a solution using one of the definitions:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 5 - (x^2-5)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 5 - x^2 + 5}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} 2x + h = 2x \end{aligned}$$

(2) Find an equation of the tangent line to the curve $y = x^2 + \ln(x)$ at the point where $x = 1$.

First we find y' so that we can determine the slope of the tangent line.

$y'(x) = 2x + \dfrac{1}{x}$. So $y'(1) = 2(1) + \dfrac{1}{1} = 3$.

Since at $x = 1$, $y(1) = 1^1 + \ln(1) = 1$, the point slope formula tells us that the equation of a line with slope 3 going through $(1, 1)$ is

$y - 1 = 3(x - 1)$.

(3) (a) If $\log_{10}(16) = 1.204$, what is an approximate value of $10^{0.301}$?

There are many ways to do this; here's one:

$$\begin{aligned} 1.204 &= \log_{10}(16) \\ 1.204 &= \log_{10}(2^4) \\ 1.204 &= 4\log_{10}(2) && \text{By a property of logarithms} \\ 0.301 &= \log_{10}(2) && \text{Divided both sides by 4} \end{aligned}$$

The last equation is the same as saying that $10^{0.301} = 2$.

(b) Find $\sec(\frac{\pi}{3})$.

$\sec(x) = \dfrac{1}{\cos(x)}$. Since $\cos(\frac{\pi}{3}) = \frac{1}{2}$, $\sec(\frac{\pi}{3}) = 1/(1/2) = 2$.

(4) Let $f(x) = e^{(x-1)} + 5$.

(a) Find a formula for f^{-1}.
Rewrite as $y = e^{(x-1)} + 5$. Switching x with y we get $x = e^{(y-1)} + 5$. To solve for y, move the 5 over to the left hand side to get $x - 5 = e^{(y-1)}$, and take $\ln(x)$ of both sides. This gives $\ln(x-5) = y - 1$, or $f^{-1}(x) = y = \ln(x-5) + 1$.

(b) What is the domain of f^{-1}? Explain.
The domain of f^{-1} is a shift to the right by 5 of the domain of $\ln(x)$, so it's $(5, \infty)$.

(5) Find the derivative of $f(x) = \ln(4x^2)\sin(x)$.

Using the product rule we get

$$f'(x) = \frac{1}{4x^2}(8x)\sin(x) + \cos(x)\ln(4x^2)$$

(6) Find the derivative of $f(x) = x^2 + 5e^{(\sin(x)+x^7)}$.

$$f'(x) = 2x + 5e^{(\sin(x)+x^7)}(\cos(x) + 7x^6)$$

(7) Find the derivative of $f(x) = \dfrac{5\cos(x) + 3^x}{2x}$.

$$f'(x) = \frac{(-5\sin(x) + 3^x\ln(3))(2x) - 2(5\cos(x) + 3^x)}{(2x)^2}$$

(8) Sketch a graph of $g(x) = 2\sin(x + \pi/2) - 1$. Indicate the scale on both the x and y axes.

Check your sketch with your calculator.

(9) Suppose that f is a function with derivative $f'(x) = x^2(x-1)$.

(a) Determine the set of points x on which f is increasing.
The roots of $f'(x)$ are at $x = 0$ and $x = 1$. To the left of 0 we see that $f'(x)$ is negative, between 0 and 1 it's still negative, and to the right of 1 it's positive. Thus $f(x)$ is increasing on the interval $(1, \infty)$.

(b) Find all the stationary points of the function f.
These are by definition the roots of f': $x = 0, 1$.

(c) Which of the stationary points are maximum, minimum, or neither? Explain.
By the first derivative test, $x = 1$ is a local minimum. There is no local max or min at $x = 0$ since f' is positive on both sides of $x = 0$.

(10) Find the equation of the tangent line to $x^2y^2 + x^2 + y^2 = 9$ at the point $(2, 1)$.

First implicitly differentiate to get

$$2xy^2 + 2yy'x^2 + 2x + 2yy' = 0$$

Solve for y':

$$2yy'x^2 + 2yy' = -2xy^2 - 2xy'(2yx^2 + 2y) = -2xy^2 - 2xy' = \frac{-2xy^2 - 2x}{2yx^2 + 2y}$$

To get the slope of the tangent line at $(2, 1)$, plug $x = 2$ and $y = 1$ into y' to get $\frac{-8}{10} = \frac{-4}{5}$. The point-slope formula now tells us that the equation of the tangent line is $y - 1 = \frac{-4}{5}(x - 2)$.

(11) If a car is 5 miles from home and is travelling at 35 mph as it begins to accelerate at a rate of 0.01 miles per hour squared, how far is he from home an hour later?

We want to find the distance function, $d(t)$. We're given that the acceleration function is $a(t) = 0.01$.

Antidifferentiating, $v(t) = 0.01t + C$. Since $v(0) = C = 35$, $C = 35$. So $v(t) = 0.01t + 35$.

To get $d(t)$ we antidifferentiate again:

$d(t) = \frac{0.01t^2}{2} + 35t + D$, for D some constant. Again, since $d(0) = 5 = D$, we see that $d(t) = 0.005t^2 + 35t + 5$.

Now we find that $d(1) = 0.005 + 35 + 5 = 40.005$ miles.