Mastery Exam Practice 3

(1) Compute the derivative of $f(x) = x^2 + 1$ using the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 1 - (x^2 - 1)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 1 - x^2 + 1}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} 2x + h = 2x \end{aligned}$$

(2) Find an equation of the tangent line to the curve $y = 2x^3 - 4x + 3$ at the point where $x = -1$.

$y' = 6x^2 - 4$. $y'(-1) = 6 - 4 = 2$. $y(-1) = -2 + 4 + 3 = 5$. The equation of the tangent line is: $y - 5 = 2(x + 1)$.

(3) (a) If $\log_{10}(2) \approx .301$, what is an approximate value of $10^{.699}$?

If $\log_{10}(2) = .301$, then $10^{.301} = 2$. Now, $10 = 10^1 = 10^{.301+.699} = 10^{.301}10^{.699} = 2 * 10^{.699}$, so $10 = 2 * 10^{.699}$, which means that $10^{.699} = 5$.

(b) Find $\tan(-\frac{\pi}{4})$.

$\tan(-\frac{\pi}{4}) = -1$.

(4) Let $f(x) = \ln(x - 1)$.

(a) Find a formula for f^{-1}.

$$\begin{aligned} y &= \ln(x-1) \\ x &= \ln(y-1) \\ e^x &= y - 1 \end{aligned}$$

$$f^{-1}(x) = e^x + 1$$

(b) What is the domain of f^{-1}? Explain.

All real numbers.

(5) Find the derivative of $f(x) = e^x x^5$.

$f'(x) = e^x x^5 + 5x^4 e^x$.

(6) Find the derivative of $f(x) = \dfrac{2\ln(x) + \sin(x)}{x^2}$.

$$f'(x) = \frac{(\frac{2}{x} + \cos(x))x^2 - 2x(2\ln(x) + \sin(x))}{x^4}.$$

(7) Find the derivative of $f(x) = \cos(x^3 + \pi) + e^{3x}$.

$f'(x) = -\sin(x^3 + \pi)(3x^2) + 3e^{3x}$.

(8) Sketch a graph of $g(x) = 3 + 3\sin(2\pi x)$. Indicate the scale on both the x and y axes.

Check on your calculator.

(9) Let $f(x) = 2x^3 + 3x^2 + 8$.

(a) Determine the set of points x on which f is increasing.

$f'(x) = 6x^2 + 6x = 6x(x + 1)$. So the stationary points of f are at $x = 0$ and $x = -1$. To the left of -1, $f' > 0$, between -1 and 0 $f' < 0$ and for $x > 0$, $f' > 0$. Thus f is increasing on $(-\infty, -1) \cup (0, \infty)$.

(b) Find all the stationary points of the function f and indicate if they are local max/min or neither.

From the above we see that the stationary points are at $x = 0, -1$, and by the first derivative test there is a local maximum at $x = -1$ and a local minimum at $x = 0$.

(10) Find the equation of the tangent line to $\ln(xy) + x^3 y = 1$ at the point $(1, 1)$.

Implicitly differentiate to get

$$\frac{1}{xy}(y + xy') + 3x^2 y + x^3 y' = 0$$

Simplify:

$$\frac{1}{x} + \frac{y'}{y} + x^3y' = -3x^2y$$

Now solve for y' to find the slope and write down the equation.

(11) Suppose $f(1) = 4$ and $f'(x) \geq 1$ for all x. If $g(x) = x^2f(2x)$, is $g'(1) \geq 12$? Explain. (Hint: Remember the speed limit law).

By the chain rule,

$$g'(x) = 2xf(2x) + x^2f'(2x)2$$

So

$$g'(1) = 2(1)f(2(1)) + (1)^2f'(2(1))2 = 2f(2) + 2f'(2)$$

Now, $f'(x) \geq 1$ for all x, and in particular $f'(2) \geq 1$. By the speed limit law,

$$f(2) - f(1) \geq 1(2-1)$$

Since $f(1) = 4$, $f(2) \geq 5$.

This tells us that

$$g'(1) \geq 2(5) + 2(1) = 12$$

And the answer is yes.