Solutions to 2.7: 6, 53, 55, 56, 62, 64, 65 and 3.1: 2, 4, 6, 8, 9, 16, 18, 22, 24

(6) If $y'(x) = 3\sin(x) - 4\cos(x)$, then $y(x) = -3\cos(x) - 4\sin(x) + C$. If $y(0) = 2$ then $2 = -3\cos(0) - 4\sin(0) + C = -3 + C$ so $C = 5$. So the solution to the IVP is $y(x) = -3\cos(x) - 4\sin(x) + 5$.

(53) If $f'(x) = 2x^3 + \sin(4x)$, then $f(x) = .5x^4 - \cos(4x)/4$. Since $f(0) = 1$, we have that $C = 5/4$.

(55) If $h''(x) = -4h(x)$, then by a fact on page 145, a general solution to this DE is $h(x) = A\sin(\sqrt{4}x) + B\cos(\sqrt{4}x)$. If $h(0) = 2$, then $2 = A\sin(2*0) + B\cos(2*0) = B$ so $B = 2$. So $h'(x) = 2A\cos(2x) - 2B\sin(2x)$. Since $h'(0) = 8$ and $B = 2$, we have that $A = 4$.

(56) We're given that $f(x) = A\cos(x) + B\sin(x)$. If $f'(\pi/3) = -1$, then $-1 = f'(\pi/3) = -A\sin(\pi/3) + B\cos(\pi/3) = -A\sqrt{3}/2 + B/2$ (*). If $f(\pi/3) = \sqrt{3}$ then $A/2 + B\sqrt{3}/2 = \sqrt{3}$ (**). From equation (*) we get that $B = A\sqrt{3} - 2$. Substituting into equation (**), we see that $A = \sqrt{3}$ and this gives that $B = 1$. Notice that here you had to solve a system of two linear equations with two unknowns.

(62) (a) $-\cos(x)$ (b) $-\cos(x)$ (c) $\sin(x)$

(64) This limit represents $\cos'(pi/2) = \sin(\pi/2) = -1$.

(65) This limit represents $\sin'(\pi) = \cos(\pi) = -1$.

(2) Let $f(x) = x^3\cos(x)$. Then letting $u(x) = x^3$ and $v(x) = \cos(x)$ so that $f(x) = u(x)v(x)$, the product rule tells us that $f' = u'v + v'u$. Now, $u'(x) = 3x^2$ and $v'(x) = -\sin(x)$, so $f'(x) = 3x^2\cos(x) + -\sin(x)x^3$.

(4) Let $h(x) = 3^{2x}$. We're asked to find the derivative using the product rule. Since $3^{2x} = 3^{x+x} = 3^x3^x$, letting $u(x) = 3^x$ and $v(x) = 3^x$, we get that $h(x) = u(x)v(x)$ and by the product rule $h' = u'v + v'u$. From section 2.6 we know that $u'(x) = v'(x) = 3^x\ln(3)$. Plugging it all into the equation we see that $h'(x) = 3^x\ln(3)3^x + 3^x3^x\ln(3) = 2\ln(3)3^{2x}$.

(6) Let $f(w) = w/(w^2+1)$. Letting $u(x) = w$ and $v(x) = w^2 + 1$, we have that $f(w) = u(w)/v(w)$ and wherever $w(x)$ is nonzero, the quotient rule gives $f' = (vu' - uv')/v^2$. Since $u'(x) = 1$ and $v'(x) = 2w$, $f'(x) = ((w^2+1)1 - w(2w))/(w^2+1)^2)$. We can simplify this further to $h'(w) = (1-w^2)/(w^2+1)$.

For part (b), notice that if $f(w) = u(w)/v(w)$, then $f(w)v(w) = u(w)$. This translates to saying that $f(w)(w^2+1) = w$. Differentiating both sides (using the product rule on the left hand side!) we see that $f'(w)(w^2+1) + 2wf(w) = 1$. We know that $f(w) = w/(w^2+1)$, and we can plug it into this equation to get $f'(w)(w^2+1) + 2w(w/(w^2+1)) = 1$. After simplifying we get that $f'(w) = ((w^2+1) - 2w^2)/(w^2+1)^2 = (1-w^2)/(w^2+1)^2$.

(8) Let $h(r) = r\ln(r)\cos(r)$. To find $h'(r)$, we use the product rule twice. First, let $u(r) = r\ln(r)$ and $v(r) = \cos(r)$. Then $h(r) = u(r)v(r)$ and $h' = u'v + v'u$ (*). We know that $v'(r) = -\sin(r)$, but we don't know $u'(r)$ since $u(r)$ is a product of two functions and we must use the product rule to find its derivative. So, let $k(r) = r$ and $l(r) = \ln(r)$. Then $u(r) = k(r)l(r)$ and $u'(r) = k'l + l'k$. Since $k'(r) = 1$ and $l'(r) = 1/r$, we get that $u'(r) = 1\ln(r) + (1/r)r = ln(r) + 1$. Plugging $u'(r)$ and $v'(r)$ into equation (*) now yields $h'(r) = (ln(r) + 1)\cos(r) + -\sin(r)r\ln(r)$.

(9) Let $h(x) = xx^{-1}$. We're asked to find the derivative in two ways, first by using the product rule, and then by simplifying the function before differentiating.

(1) If $u(x) = x$ and $v(x) = x^{-1}$, then $h' = u'v+v'u$. We know that $u'(x) = 1$ and $v'(x) = -1x^{-2}$. Then $h'(x) = x^{-1} + x(-x^{-2}) = x^{-1} - x^{-1} = 0$.

(2) $h(x) = xx^{-1} = x(1/x) = 1$, so $h'(x) = 0$.

(16) Same instructions as in the previous problem, but this time we're going to use the quotient rule. Let $h(x) = x/\sqrt{x}$.

(1) $h(x) = u(x)/v(x)$ where $u(x) = x$ and $v(x) = \sqrt{x}$. By the quotient rule, $h' = (vu' - uv')/v^2$. We know that $u'(x) = 1$ and $v'(x) = -1/\sqrt{x}$, so $h'(x) = (\sqrt{x}1 - x(-1/\sqrt{x}))/(\sqrt(x)^2) = (\sqrt{x} + x/\sqrt{x})/x = 1/(2\sqrt{x})$. (Here the nontrivial manipulation is to notice that $x/\sqrt{x} = \sqrt{x}$.)

(2) $h(x) = x/\sqrt{x} = \sqrt{x}$, so $h'(x) = 1/(2\sqrt{x})$. (Here we use the fact that $\sqrt{x} = x^{1/2}$ and apply our usual rule for taking derivatives of such functions).

(18) $h(x) = \frac{1-x^2}{1+x}$.

(1) Let $u(x) = 1 - x^2$ and $v(x) = 1 + x$. Then $u'(x) = -2x$ and $v'(x) = 1$. Using the quotient rule, $h'(x) = (vu' - uv')/v^2 = ((1 + x)(-2x) - (1 - x^2)1)/(1 + x)^2 = -(x^2 + 2x + 1)/(1 + x)^2 = -1$.

(2) $h(x) = \frac{1-x^2}{1+x} = \frac{(1+x)(1-x)}{1+x} = 1 - x$, so $h'(x) = -1$.

(22) Let $f(x) = x^2\cos(x)$. Let $u(x) = x^2$ and $v(x) = \cos(x)$. Then $u'(x) = 2x$ and $v'(x) = -\sin(x)$. The product rule tells us that $f' = u'v + v'u = 2x\cos(x) - \sin(x)x^2$.

(24) Let $f(x) = x^3e^x$. Let $u(x) = x^3$ and $v(x) = e^x$. Then $u'(x) = 3x^2$ and $v'(x) = e^x$, so $f' = u'v + v'u = 3x^2e^x + x^3e^x = e^x(3x^2 + x^3)$.