1 Introduction

I hope that this exposition will clarify some of the confusion about Taylor Polynomials. To give you a big picture, the fact that we can approximate functions using polynomials is a fundamental theme in all of higher mathematics - some people make their living by abstracting this notion.

Let's recall the definition of a polynomial function:

Definition 1.1. *A polynomial $p(x)$ of degree n is a function of x that has the form*

$$p(x) = a_0 + a_1 x + a_1 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n$ are fixed real numbers.

For example, $p(x) = 1 + 2x - \pi x^3$ is a polynomial of degree 3. Here $a_0 = 1, a_1 = 2, a_2 = 0$, and $a_3 = \pi$.

2 Why Polynomials?

(*) Evaluating them at any number is very easy.

After all, polynomials are functions that are constructed by only adding and multiplying our input variable. For example, if $f(x) = 2 + x^2$, then $f(1) = 2 + (1)^2 = 3$.

(*) Derivatives of polynomials are again polynomials, and we know their formula (it is given by the "power rule" for taking derivatives).

For example, we all know that if $f(x) = x + 3x^2$, then $f'(x) = 1 + 6x$, and $f''(x) = 6$. We also see that since every polynomial has a derivative, and a derivative of a polynomial is again a polynomial, then polynomials have derivatives of all order (first, second, third, etc.,)

(*) The same is true for anti-derivatives of polynomials: they exist and we know their formula.

For example, recall that if $f(x) = x + x^2$, then $F(x) = \frac{x^2}{2} + \frac{x^3}{3}$ and $F'(x) = f(x)$.

Unfortunately, not all functions are polynomials. For example, although it would take some time to prove this, $\sin(x)$ is not a polynomial function (can you find real numbers $a_0, a_1, \ldots, a_n$ such that $\sin(x) = a_0 + a_1 x + \cdots a_n x^n$?).

3 Connecting polynomials to other functions

Let's start with a bit of an analogy: we can all agree that a house can be a very complicated structure built from "simple" bricks. Knowing that a house is built from bricks won't necessarily tell us everything about the house, but it might be good enough to study properties such as heat loss or weather resistance.

Another way to say it is that even though "the whole is greater than the sum of its parts," the sum of the parts can still tell you something useful about the "whole."

This is precisely the role that polynomials play in connection with other functions: almost every function that we want to study can be built from polynomials in such a way that these polynomials will tell us a little about the value of the function on some interval and the function's derivatives - to a certain extent.

To discover Taylor polynomials, we ask the following question (on the left is the general statement, and on the right is the special case of $f(x) = \ln(x)$, $x_0 = 1$, and $n = 2$).

Given a "nice" function f, a point x_0 in f's domain, and a natural number n, does there exist a polynomial $T_n(x)$ such that	Given $f(x) = \ln(x)$, and $x_0 = 1$, does there exist a polynomial $T_2(x)$ of degree 2 such that
$T_n(x_0) = f(x_0)$	$T_n(1) = \ln(1)$
$T_n'(x_0) = f'(x_0)$	$T_n'(1) = \ln'(1)$
$T_n''(x_0) = f''(x_0)$	$T_n''(x_0) = \ln''(1)$
$\vdots$	We only go up to $n = 2$ in this example, so nothing here
$T_n^{(n)}(x_0) = f^{(n)}(x_0)$	

In plain English, if you give me a function f and a point x_0 in f's domain, can I find a polynomial of any degree that I want with the property that this polynomial's derivatives at x_0 agree with the derivatives of f at x_0 for the first n derivatives?

We know that the answer is yes, but let's see how we give the construction.

(*) **Case** $n = 0$

Suppose in the above question $n = 0$. Then, does there exist a polynomial $T_0(x)$ of degree 0 (which is just a constant), such that

(1) $T_0(x_0) = f(x_0)$

Yes! Define $T_0(x) = f(x_0)$. Then $T_0(x_0)$ is equal to $f(x_0)$, and $T_0(x)$ is a polynomial of degree 0 (a constant polynomial).

In our example, $f(x) = \ln(x)$, $x_0 = 1$, and $\ln(x_0) = \ln(1) = 0$. Thus $T_0(x) = 0$.

(*) **Case** $n = 1$

Now we want to find a polynomial $T_1(x)$ of degree 1 such that

(1) $T_1(x_0) = f(x_0)$
(2) $T_1'(x_0) = f'(x_0)$

Let's try $T_1(x) = f(x_0) + f'(x_0)(x - x_0)$.

The first condition is satisfied since $T_1(x_0) = f(x_0) + f'(x_0)(x_0 - x_0) = f(x_0)$.

For the second condition, we must find $T_1'(x)$. But this is easy since $T_1(x)$ is a polynomial: $T_1'(x) = f'(x_0)$. The little magic that happened there was noticing that $T_1(x) = f(x_0) + f'(x_0)(x - x_0) = f(x_0) + f'(x_0)x - f'(x_0)x_0$, and since $f'(x_0)$ and x_0 is just some number the result follows from the power rule.

In our example, $T_1(x) = \ln(1) + \ln'(1)(x - 1)$. Since $\ln'(x) = 1/x$, we see that $\ln'(1) = 1$ and $T_1(x) = 0 + 1(x - 1) = (x - 1)$.

(*) **General case**

In general, the Taylor polynomial of degree n around x_0 will have the form

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

You can easily check that it satisfies the properties that we listed in our question.

4 Summary and an example

To summarize, Taylor polynomials of a function f give approximate f *near some point* (That's why this x_0 keeps popping up: You wouldn't expect the tangent line to the sin curve at π to be a good approximation for the values of sin around $\frac{\pi}{2}$).

Exercise 4.1. *Find the third degree Taylor polynomial for* $\sin(x)$ *about* $x_0 = \frac{\pi}{2}$.

To find the third degree approximation, we need to go up to the third derivative: $\sin'(x) = \cos(x), \sin''(x) = -\sin(x)$, and $\sin'''(x) = -\cos(x)$. Evaluating these at $\frac{\pi}{2}$ we get: $\sin(\frac{\pi}{2}) = 1$, $\sin'(\frac{\pi}{2}) = 0$, $\sin''(\frac{\pi}{2}) = -1$, and $\sin'''(\frac{\pi}{2}) = 0$.

Now we plug into our formula:

$$\begin{array}{ccccccccc}
T_3(x) = & f(x_0) & + & f'(x_0)(x-x_0) & + & \frac{f''(x_0)}{2}(x-x_0)^2 & + & \frac{f'''(x_0)}{3!}(x-x_0)^3 \\
 & \| & & \| & & \| & & \| \\
T_3(x) = & \sin(\frac{\pi}{2}) & + & \sin'(\frac{\pi}{2})(x-\frac{\pi}{2}) & + & \frac{\sin''(\frac{\pi}{2})}{2}(x-\frac{\pi}{2})^2 & + & \frac{\sin'''(\frac{\pi}{2})}{3!}(x-\frac{\pi}{2})^3 \\
 & \| & & \| & & \| & & \| \\
T_3(x) = & 1 & + & 0 & + & -\frac{1}{2}(x-\frac{\pi}{2})^2 & + & 0
\end{array}$$

So we get that $T_3(x) = 1 - \frac{(x-\frac{\pi}{2})^2}{2}$, and we can use this function to find approximate values of $\sin(x)$ near $\frac{\pi}{2}$.