METRIC DIMENSIONS AND TAMENESS IN EXPANSIONS OF THE REAL FIELD

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ABSTRACT. For first-order expansions of the field of real numbers, nondefinability of the set of natural numbers is equivalent to equality of topological and Assouad dimension on images of closed definable sets under definable continuous maps.

1. INTRODUCTION

We investigate relations between metric dimensions in real euclidean spaces and definability in expansions of $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot, (r)_{r \in \mathbb{R}})$, the real field with constants for all real numbers. The intended audience includes both mathematicians and logicians. The main result is a dichotomy: Roughly, any given expansion of $\overline{\mathbb{R}}$ either defines all real projective sets or there is a striking agreement of various dimensions on an important subcollection of the definable sets.

For readers familiar with basic first-order logic (FOL, for short), structures in a given language, and expansions thereof, are defined as usual (see, *e.g.*, Marker [24, 1.4.15]). As the language of $\overline{\mathbb{R}}$ contains a constant for each real number, there is no issue of whether "definable" means "with parameters" or "without parameters" in expansions of $\overline{\mathbb{R}}$. We identify interdefinable structures except when dealing with syntactic notions such as model completeness.

We ultimately refer readers who are not familiar with FOL to van den Dries and Miller [8] for an essentially FOL-free introduction to definability theory over $\overline{\mathbb{R}}$. However, we do give here some needed definitions and conventions. First, a **structure** on a set X is a sequence $\mathfrak{S} := (\mathfrak{S}_m)_{m=1}^\infty$ such that for each m:

   (1) $\mathfrak{S}_m$ is a boolean algebra of subsets of X^m (the m-fold cartesian power of X).
   (2) If $S \in \mathfrak{S}_m$, then $S \times X, X \times S \in \mathfrak{S}_{m+1}$.
   (3) If $S \in \mathfrak{S}_{m+1}$, then the projection of S on the first m coordinates is in $\mathfrak{S}_m$.
   (4) If $1 \leq i \leq j \leq m$, then $\{ x \in X : x_i = x_j \} \in \mathfrak{S}_m$.

A **structure on $\overline{\mathbb{R}}$** is a structure on $\mathbb{R}$ such that (the graphs of) addition and multiplication belong to $\mathfrak{S}_3$ and every real singleton $\{r\}$ belongs to $\mathfrak{S}_1$. (In the terminology of [8], a structure on $\overline{\mathbb{R}}$ is a structure on $(\mathbb{R}, +, \cdot)$ such that $\mathfrak{S}_1$ contains every real singleton.) In

this paper, we deal almost exclusively with structures on $\overline{\mathbb{R}}$, but it is often useful to know that there is wider context.

We ask readers not familiar with FOL to take on faith that for every expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$ in the sense of FOL there is a structure $\mathfrak{S}$ ($:= \mathfrak{S}(\mathfrak{R})$) on $\overline{\mathbb{R}}$ such that each $\mathfrak{S}_m$ consists of the subsets of $\mathbb{R}^m$ that are definable in $\mathfrak{R}$, and for every structure $\mathfrak{S}$ on $\overline{\mathbb{R}}$ there is an expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$ in the sense of FOL such that each $\mathfrak{S}_m$ consists of the subsets of $\mathbb{R}^m$ that are definable in $\mathfrak{R}$. Thus, for the purposes of this paper, readers can take "expansion of $\overline{\mathbb{R}}$" to mean "structure on $\overline{\mathbb{R}}$" and definability to mean belonging to the appropriate $\mathfrak{S}_m$. (It is crucial to understand that the notion of definability must always be taken with respect to some given structure.) We regard maps as set-theoretic objects, so to say that a map $\mathbb{R}^m \supseteq X \to \mathbb{R}^n$ is definable is to say that its graph is definable.

Expansions of $\overline{\mathbb{R}}$ can be partially ordered by setting $\mathfrak{R} \leq \mathfrak{R}'$ if every set definable in $\mathfrak{R}$ is definable in $\mathfrak{R}'$. There is a maximal element: Take each $\mathfrak{S}_m$ to be the power set of $\mathbb{R}^m$. By definition, $\overline{\mathbb{R}}$ is a minimal element. The subsets of $\mathbb{R}^m$ definable in $\overline{\mathbb{R}}$ consist of the **semialgebraic** sets in $\mathbb{R}^m$, that is, finite unions of sets of the form

$$\{\, x \in \mathbb{R}^m : f(x) = 0, g_1(x) > 0, \ldots, g_j(x) > 0 \,\}$$

where $f, g_1, \ldots, g_j \in \mathbb{R}[x_1, \ldots, x_m]$; *cf.* [8, 2.5.3]. We say that $\mathfrak{R}$ and $\mathfrak{R}'$ are **interdefinable** if $\mathfrak{R} \leq \mathfrak{R}'$ and $\mathfrak{R}' \leq \mathfrak{R}$, equivalently, if $\mathfrak{S}(\mathfrak{R}) = \mathfrak{S}(\mathfrak{R}')$.

For any expansion $\mathfrak{R}$ of $\overline{\mathbb{R}}$ and sequence $\mathcal{A} := (\mathcal{A}_m)_{m=1}^\infty$ of collections $\mathcal{A}_m$ of subsets of $\mathbb{R}^m$ there is a minimal $\mathfrak{R}' \geq \mathfrak{R}$ such that, for every m, every $A \in \mathcal{A}_m$ is definable in $\mathfrak{R}'$. Rather than declare notation for this level of generality, we shall use various ways to indicate expansions of $\mathfrak{R}$ by sets of interest. For example, if $E \subseteq \mathbb{R}^n$, then we let $(\mathfrak{R}, E)$ denote a minimal $\mathfrak{R}'$ that defines E as well as every set definable in $\mathfrak{R}$.

The reader should now be able to apply [8] here as needed (begin with its second section and first two appendices).

What can be said about the lattice (up to interdefinability) of all expansions of $\overline{\mathbb{R}}$? For reasons that we shall not attempt to explain here, we regard this question as too vague at best and intractable at worst. However, there are more reasonable versions, as we now begin to explain.

The real projective hierarchy. Consider $(\overline{\mathbb{R}}, \mathbb{N})$, the expansion of $\overline{\mathbb{R}}$ by the set of non-negative integers $\mathbb{N}$. Logicians will immediately observe that $\mathrm{Th}(\mathbb{R}, +, \cdot, \mathbb{N})$ is undecidable. But much more is true: $(\overline{\mathbb{R}}, \mathbb{N})$ defines every real Borel set (see, *e.g.*, Kechris [21, (37.6)]), hence also every real projective set in the sense of descriptive theory ([21, Chapter V]). Thus, the definable sets of $(\overline{\mathbb{R}}, \mathbb{N})$ comprise the real projective hierarchy. To put this another way, the study of the definable sets of $(\overline{\mathbb{R}}, \mathbb{N})$ is essentially classical descriptive set theory. Thus, even set-theoretic independence can arise from seemingly-innocent questions such as whether every set definable in $(\overline{\mathbb{R}}, \mathbb{N})$ is Lebesgue measurable (Solovay [29]). Because of such complications, nondefinability of $\mathbb{N}$ is generally regarded as necessary for the definability theory of any particular expansion of $\overline{\mathbb{R}}$ to be analyzable by model-theoretic methods.

The question arises: What can be said about expansions of $\overline{\mathbb{R}}$ that do not define $\mathbb{N}$? This question underlies the "tameness program over $\overline{\mathbb{R}}$"; see the introduction of [26] for a more detailed exposition. The main result of this paper (Theorem A, below) is one answer, but we need a few more preliminaries for its statement.

Unless indicated otherwise, the variables j, k, l, m, n, p range over $\mathbb{N}$. Given a set X, its power set is denoted by $\mathcal{P}(X)$ and its n-fold cartesian power by X^n, with X^0 the one-point set $\{0\}$. For convenience, we often identify $X^m \times X^n$ with X^{m+n}, and $(X^m)^n$ with X^{mn}; in particular, $X^m \times X^0 \cong X^0 \times X^m \cong X^m$. By a **coordinate projection** of $S \subseteq X^n$ we mean the image of S under a map

$$(x_1, \ldots, x_n) \mapsto \left(x_{\lambda(1)}, \ldots, x_{\lambda(m)}\right) : X^n \to X^m$$

where $0 \leq m \leq n$ and $\lambda \colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ is strictly increasing. We usually say just "projection" instead of "coordinate projection".

If X is regarded as a subset of some topological space, then we denote its interior by $\operatorname{int} X$ and closure by $\operatorname{cl} X$. We say that X: **has interior** if $\operatorname{int} X \neq \emptyset$; **has no interior** if $\operatorname{int} X = \emptyset$; and is **constructible** if it is a boolean combination of open sets. If X is given a topology, then X^n is always regarded in the product topology.

Given a topological space X, we define a sequence of functions

$$\dim_X := \left(\dim_{X,m} \colon \mathcal{P}(X^m) \to \mathbb{N} \cup \{-\infty\}\right)_{m \in \mathbb{N}}$$

by letting $\dim_{X,m} A$ (for $A \subseteq X^m$) be the supremum of all $k \in \mathbb{N}$ such that some projection of A on X^k has interior. Observe that $\dim_{X,m} \emptyset = -\infty$ for all $m \in \mathbb{N}$. We usually suppress the subscripts, writing just $\dim$ instead of $\dim_{X,m}$. (Some ambiguity can then arise—e.g., if $X \neq \emptyset$, then $\dim_{X,2} X^2 = 2$ and $\dim_{X^2,1} X^2 = 1$—but intent should always be clear from context.) We also tend to shorten "$\dim \operatorname{cl}$" to "dimcl". Our use of "$\dim$" here conflicts with some of the dimension theory literature but is consistent with some key model-theoretic sources that we cite ($\dim$ is often useful when dealing with structures on X). Some properties (proofs are exercises):

(1) If $A \subseteq X^m$, then $\dim A = m$ if and only if A has interior.
(2) $\dim$ is **increasing**; i.e., $\dim A \leq \dim B$ for all $m \in \mathbb{N}$ and $A \subseteq B \subseteq X^m$.
(3) $\dim$ is **stable** on compact sets; i.e., $\dim(A \cup B) = \max(\dim A, \dim B)$ for all $m \in \mathbb{N}$ and compact $A, B \subseteq X^m$.
(4) $\dim$ is **logarithmic**, that is, $\dim(A \times B) = \dim A + \dim B$ for all $A \subseteq X^m$ and $B \subseteq X^n$.
(5) If $A \subseteq X$, then $\dim A = 0$ if and only if A is nonempty and has no interior.
(6) $\dim$ is stable on constructible subsets of X.
(7) If X is a metric space, then $\dim \leq \dim_{\mathrm{H}}$, where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension.

(Hint for (6): constructible sets either have interior or are nowhere dense. Hint for (7): $\dim_{\mathrm{H}}$ cannot increase under Lipschitz maps.)

Some global assumptions:

— we regard $\mathbb{R}^n$ in the usual box topology, but we also employ the usual notation and conventions for working with the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$;
— E denotes an arbitrary subset of an arbitrary $\mathbb{R}^n$ unless otherwise indicated (e.g., by explicitly writing "$n = 1$" or "$E \subseteq \mathbb{R}$");
— $\mathfrak{R}$ denotes a fixed, but arbitrary, expansion of $\overline{\mathbb{R}}$;
— "definable" means "definable in $\mathfrak{R}$" unless indicated otherwise (possibly only by context).

Many tameness conditions on $\mathfrak{R}$ imply that every definable subset of $\mathbb{R}$ either has interior or is nowhere dense, a condition that yields many desirable consequences for the definable

sets; see [26]. It would take us too far afield to go into details here, but we do need one basic result now:

1.1 (see [26, §7, Main Lemma]). *The following are equivalent.*
   — *Every definable subset of $\mathbb{R}$ has interior or is nowhere dense.*
   — $\dim = \mathrm{dimcl}$ *on all definable subsets of $\mathbb{R}$.*
   — $\dim = \mathrm{dimcl}$ *on all definable sets.*
   — *Every definable set either has interior or is nowhere dense.*

Thus, the technical dimensional condition "$\dim = \mathrm{dimcl}$ on all definable sets" corresponds to a natural tameness condition on $\mathfrak{R}$, namely, that each definable set either have interior or be nowhere dense. Nevertheless, whatever good properties $\dim$ or dimcl might have, they cannot be regarded as interesting *topological* dimensions because $\dim X = 1$ for each fixed nonempty topological space X. This leads naturally to wondering what can be said about $\mathfrak{R}$ under assumptions on various topological or metric dimensions.

One classical dimension for topological spaces X is the **small inductive dimension**, $\mathrm{ind}\, X$, defined inductively by: $\mathrm{ind}\, \emptyset = -\infty$; for $X \neq \emptyset$, $\mathrm{ind}\, X$ is the infimum of all k such that, for every $x \in X$ and open neighborhood V of x, there is an open U such $x \in \mathrm{cl}\, U \subseteq V$ and the boundary of U (regarded as a topological space via the subspace topology) has $\mathrm{ind} < k$. (Often, $\mathrm{ind}\, \emptyset$ is defined to be -1, but we prefer $-\infty$ for technical reasons.) The definition is then extended to subsets of X by passing to the subspace topology. As with $\dim$, we tend to write indcl instead of $\mathrm{ind}\,\mathrm{cl}$. Although defined for every topological space, ind is not particularly well behaved on all topological spaces, nor in this generality does it bear any fixed relation (always $\leq$, always $=$, or always $\geq$) to certain other reasonable notions of topological dimension. But heuristically, all reasonable notions of topological dimension coincide on separable metrizable spaces (for more precision, see Engelking [9], our primary reference for topological dimension theory). Thus, when working with subsets of $\mathbb{R}^n$, we are justified in regarding ind as *the* topological dimension.

The relation between ind and $\dim$ can be subtle, even in real n-spaces. While it is easy to see that $\mathrm{ind} = \dim$ on all subsets of $\mathbb{R}$, we have only $\mathrm{ind} \leq \dim$ on all subsets of $\mathbb{R}^2$ (with inequality possible, even on compact sets), and each of $\dim < \mathrm{ind}$, $\dim = \mathrm{ind}$ and $\dim > \mathrm{ind}$ occur even among the G_δ subsets of $\mathbb{R}^3$. On the other hand, $\mathrm{ind} \leq \dim$ on all F_σ sets; in particular, $\mathrm{indcl} \leq \mathrm{dimcl}$ always. (See 2.3 for more details.) It is thus natural to wonder what can be said about $\mathfrak{R}$ if either $\mathrm{ind} = \dim$ on all definable sets or $\mathrm{ind} = \mathrm{indcl}$ on all definable sets (*cf.* 1.1). We shall indeed produce some answers here, but because we are thinking about $\mathbb{R}$ as a metric space, we prefer to bypass this level of generality in favor of "dimensional coincidences" of $\dim$ or ind with various metric dimensions. We do have a particular metric dimension in mind, called **Assouad dimension** by some, but we postpone (to Section 4) giving the definition as we think its technical nature would only be distracting at this point; for now, we only list enough of its basic properties over real n-spaces to justify our interest.

1.2. There is a sequence $\mathrm{Dim} := \big(\mathrm{Dim}_m \colon \mathcal{P}(\mathbb{R}^m) \setminus \{\emptyset\} \to [0, \infty)\big)_{m \in \mathbb{N}}$ such that for all $m \in \mathbb{N}$:
   (1) $\mathrm{Dim}_m = m$ on nonempty open subsets of $\mathbb{R}^m$.
   (2) Dim_m is increasing.
   (3) Dim_m is stable on closed sets.

4

(4) $\mathrm{Dim}_p f(A) = \mathrm{Dim}_m A$ for all $A \subseteq \mathbb{R}^m$, $p \in \mathbb{N}$ and bi-Lipschitz maps $f \colon A \to \mathbb{R}^p$ (**Lipschitz invariance**).

(5) $\mathrm{Dim}_m = \mathrm{Dim}_m \, \mathrm{cl}$ (**closure invariance**).

(6) $\mathrm{Dim}_m \geq \overline{\dim}_{\mathrm{M}}$ on bounded sets, where $\overline{\dim}_{\mathrm{M}}$ denotes upper Minkowski dimension.

(7) $\mathrm{Dim}_1\{\, 1/k : 0 \neq k \in \mathbb{N} \,\} = 1$.

As with dim, we tend to suppress the subscripted m. We shall also have $\mathrm{Dim}\,\emptyset = -\infty$.

The reader need not yet know the definition of $\overline{\dim}_{\mathrm{M}}$ (also given in Section 4); its appearance here is only to lend credence to an empirical heuristic: *If $E \neq \emptyset$ and $\mathrm{ind}\,E = \mathrm{Dim}\,E$, then all dimensions commonly encountered in geometric measure theory, fractal geometry and analysis on metric spaces are equal on E.*[*] We refer the reader to Luukkainen [23] for history and explanations of technical significance (but there, ind is denoted by dim, Dim by $\dim_{\mathrm{A}}$, and $\overline{\dim}_{\mathrm{M}}$ by $\overline{\dim}_{\mathrm{B}}$). Further information on metric dimension theory can be found in Falconer [10] and Mattila [25] (but there is no mention of Assouad dimension, and again, notation and conventions vary).

Remarks. (a) We might reasonably demand that the first four properties of 1.2 hold in order for a sequence of functions $\bigl(\mathcal{P}(\mathbb{R}^m) \setminus \{\emptyset\} \to [0,\infty)\bigr)_{m \in \mathbb{N}}$ to be considered as a system of metric dimensions for real euclidean spaces, but preservation under closure (which renders superfluous the phrase "on closed sets" in the third property) fails for $\dim_{\mathrm{H}}$. (b) It is known that $\overline{\dim}_{\mathrm{M}}$ satisfies the first five properties on bounded sets, but $\overline{\dim}_{\mathrm{M}}\{\, 1/k : 0 \neq k \in \mathbb{N} \,\} = 1/2$. (c) If $E \neq \emptyset$ and $\dim E = \mathrm{Dim}\,E$, then heuristically, all *metric* dimensions should agree on E. (It is a routine consequence of $\overline{\dim}_{\mathrm{M}} \leq \mathrm{Dim}$ on bounded sets that $\dim_{\mathrm{H}} \leq \mathrm{Dim}$, and recall that $\dim \leq \dim_{\mathrm{H}}$.)

We are now ready to state our main result:

Theorem A. *If $\mathfrak{R}$ does not define $\mathbb{N}$, then $\mathrm{ind} = \mathrm{Dim} = \dim$ on projections of closed definable sets.*

An equivalent local version:

Theorem A′. *If E is closed, $f \colon E \to \mathbb{R}^p$ is continuous, and $(\overline{\mathbb{R}}, f)$ does not define $\mathbb{N}$, then $\mathrm{ind}\, f(E) = \mathrm{Dim}\, f(E) = \dim f(E)$.*

(By basic definability, the conclusion of Theorem A is equivalent to $\mathrm{ind} = \mathrm{Dim} = \dim$ on images of closed definable sets under definable continuous maps.) Heuristically: *In order for the definability theory of $\mathfrak{R}$ to be amenable to model-theoretic methods, we must have $\mathrm{ind} = \mathrm{Dim} = \dim$ at least on all images of closed definable sets under definable continuous maps.*

It is easy to see that the converse of Theorem A holds, indeed, if either $\mathrm{Dim} = 0$ on all countable compact definable subsets of $\mathbb{R}$ or $\mathrm{ind} \geq \dim$ on all compact definable subsets of $\mathbb{R}^2$, then $\mathfrak{R}$ does not define $\mathbb{N}$. For the former, recall 1.2.5 and 1.2.7; for the latter, see 2.4 below. We now collect a few easy applications of Theorem A to the tameness program.

1.3. **Corollary.** *If $\mathrm{ind}\,E \neq \mathrm{Dim}\,E$, then either $\mathrm{ind}\,E \neq \mathrm{ind}\,\mathrm{cl}\,E$ or $(\overline{\mathbb{R}}, E)$ defines $\mathbb{N}$.*

Proof. If $(\overline{\mathbb{R}}, E)$ does not define $\mathbb{N}$, then $\mathrm{ind}\,\mathrm{cl}\,E = \mathrm{Dim}\,\mathrm{cl}\,E$ by Theorem A. Recall that $\mathrm{Dim} = \mathrm{Dim}\,\mathrm{cl}$. $\qquad\square$

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[*]Conventions for dealing with dimensions of $\emptyset$ vary in the literature.

Heuristically: *If any of the commonly-encountered metric dimensions fail to coincide on E, it is only because E is* topologically *noisy or it encodes enough information to define (over $\overline{\mathbb{R}}$) all real projective sets.*

We can add another condition to 1.1:

1.4. **Corollary.** *Every definable subset of $\mathbb{R}$ has interior is nowhere dense if and only if* dim = Dim *on all definable sets.*

Proof. Suppose that every definable subset of $\mathbb{R}$ has interior is nowhere dense. Then $\mathfrak{R}$ does not define $\mathbb{Q}$, hence neither $\mathbb{N}$, so dimcl = Dimcl = Dim on all definable sets. By 1.1, we also have dimcl = dim on all definable sets. Hence, dim = Dim on all definable sets.

The converse is immediate from Dimcl = Dim. □

We can now address some issues raised earlier:

1.5. **Corollary.** *The following are equivalent.*
  (1) ind = Dim *on all definable sets.*
  (2) ind = indcl *on all definable sets.*
  (3) ind = dim = Dim *on all definable sets.*
  (4) ind = dim *on all definable sets.*
  (5) ind $\geq$ dim *on all definable sets.*

Proof. (1)$\Rightarrow$(2) is immediate from Dimcl = Dim.

(2)$\Rightarrow$(3). If ind = indcl on all definable sets, then $\mathfrak{R}$ does not define $\mathbb{Q}$, hence neither $\mathbb{N}$, so indcl = Dimcl = dimcl on all definable sets by Theorem A. Thus, ind = Dim = dimcl on all definable sets. Since ind = dim on all subsets of $\mathbb{R}$, we also have dim = dimcl on all definable sets by 1.1, and so ind = Dim = dim on all definable sets.

(5)$\Rightarrow$(1). Assume that ind $\geq$ dim on all definable sets. As mentioned earlier, indcl $\leq$ dimcl always, so dim $\leq$ ind $\leq$ dimcl on all definable sets. Hence, it suffices by 1.1 and 1.4 to show that every definable subset of $\mathbb{R}$ has interior or is nowhere dense; if not, there would be a definable subset of $\mathbb{R}$ whose characteristic function has ind = 0 and dim = 1. □

We do not know whether the above are also equivalent to dimcl = dim on all definable sets, but we do have a partial answer:

1.6. **Corollary.** *If $\mathfrak{R}$ defines no Cantor subsets of $\mathbb{R}$ and* dim = dimcl *on all definable subsets of $\mathbb{R}$, then* ind = Dim = dim *on all definable sets.*

(By "Cantor" we mean nonempty, compact, totally disconnected, and no isolated points.)

Proof. By 1.5, it suffices to show that dim $\leq$ ind on all definable sets. It is an exercise that the assumption is equivalent to: Every nonempty definable subset of $\mathbb{R}$ has interior or an isolated point. Suppose that E is definable. By [26, 3.4], after permutation of coordinates there is a box $U \subseteq \mathbb{R}^{\dim E}$ and a continuous map $U \to \mathbb{R}^{n-\dim E}$ whose graph is contained in E. Hence, $\dim E \leq \operatorname{ind} E$. □

There are expansions of $\overline{\mathbb{R}}$ that define Cantor sets and every definable subset of $\mathbb{R}$ has interior or is nowhere dense; see Friedman *et al.* [15], and [20]. We do not know whether any of them satisfy dim $\leq$ ind on all definable sets.

It is immediate from Theorem A that if every definable set is a projection of a closed definable set, then ind = Dim = dim on all definable sets. We can do better:

1.7. Corollary. *If every definable subset of $\mathbb{R}$ is F_σ, then* $\mathrm{ind} = \mathrm{Dim} = \dim$ *on all definable sets.*

Proof. By 1.6, it suffices to show that then every definable subset of $\mathbb{R}$ has interior or is nowhere dense, and no Cantor subsets of $\mathbb{R}$ are definable.

Let $A \subseteq \mathbb{R}$ be definable and dense in a nonempty open interval I. Since both $I \cap A$ and $I \setminus A$ are definable, they are each F_σ by assumption. As $I \setminus A$ has no interior, it is meager (in the sense of Baire). Hence, $I \cap A$ is nonmeager, and thus has interior.

If $A \subseteq \mathbb{R}$ is a Cantor set, then the complement in A of the set of left endpoints of the complementary intervals of A is definable (exercise) and not F_σ (because it is both comeager and codense in the closed set A). $\qquad\square$

In particular,

1.8. Corollary. *If every definable subset of $\mathbb{R}$ has interior or is countable, then* $\mathrm{ind} = \mathrm{Dim} = \dim$ *on all definable sets.*

There are many examples of $\mathfrak{R}$ such that every definable subset of $\mathbb{R}$ has interior or is countable. Indeed, there are many examples such that every definable subset of $\mathbb{R}$ has interior or is finite (o-minimality)—see [8] for just a few—but there are also examples that define countably infinite sets; for some, see van den Dries [6], Friedman and Miller [16, 17], Miller and Tyne [28], [26], and 5.1 below.

Two important precursors to this paper should be mentioned even though there is no formal dependence. (i) In joint work with Fornasiero [13], we established that if $E \subseteq \mathbb{R}$ is nonempty, bounded and nowhere dense, then $(\overline{\mathbb{R}}, E)$ avoids defining $\mathbb{N}$ if and only if $\overline{\dim}_{\mathrm{M}} E = 0$. Ideas from the proof[†] play a crucial role in the proof of Theorem A. (ii) Fornasiero announced in [12] a weaker version of Theorem A, namely, that it holds with $\dim_{\mathrm{H}}$ in place of Dim (subject to the usual proviso that $\dim_{\mathrm{H}} \emptyset$ is often defined to be 0). Thus, it would suffice for us here to show just that $\dim_{\mathrm{H}} = \mathrm{Dim}$ on all projections of nonempty closed definable sets if $\mathfrak{R}$ does not define $\mathbb{N}$. But we have not been able to accomplish this, as we do not know of any appropriately useful criteria for checking $\dim_{\mathrm{H}} = \mathrm{Dim}$. Neither does Fornasiero's proof generalize to Dim as it relies on some properties of $\dim_{\mathrm{H}}$ that are known to fail for Dim. On the other hand, we could appeal to [12] for the "ind = dim" part of Theorem A, but this is relatively easy; we shall give our own proof (2.15) as a natural step toward establishing Theorem A.

Here is an outline of the rest of this paper. Section 2 consists of some technical preliminaries, mostly topological, culminating in the proof of the "ind = dim" part of Theorem A. While the proof of the "dim = Dim" part of Theorem A requires an actual definition of Dim, the cases $n = 1$ and $\dim = 0$ require only two easily-stated properties beyond those of 1.2, and the proofs illustrate some key ideas needed for the proof of Theorem A as a whole. Hence, with an eye toward possible generalization, we present in Section 3 an axiomatic proof of these basic cases. We also dispose of the trivial (relative to existing technology) case that $\mathfrak{R}$ defines no infinite discrete closed subsets of $\mathbb{R}$. In Section 4, we give the definitions of Dim and $\overline{\dim}_{\mathrm{M}}$, and prove Theorem A via a more technical result (Theorem B). We conclude in Section 5 with a few remarks.

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[†]One of which, as noted in [13], was communicated to Miller by K. Falconer in personal correspondence. It was also Falconer who first drew our attention to Assouad dimension.

2. Preliminaries

Let X, Y be sets. Given $f \colon X \to Y$ and $S \subseteq X$, we let $f{\restriction}S$ denote the restriction of f to S. We identify a function $f \colon X^0 \to Y$ with the constant $f(0) \in Y$. Given $Z \subseteq X \times Y$ and $x \in X$, we put $Z_x := \{\, y \in Y : (x, y) \in Z \,\}$; this notation will be used primarily in settings where both X and Y are finite cartesian powers of $\mathbb{R}$, but it can also be used to resolve any potential ambiguity caused by the usual kinds of subscripting of indices that arise in mathematical exposition. If Y is a product $Y_1 \times Y_2$ and $(x, u) \in X \times Y_1$, we tend to abbreviate $Z_{(x,u)}$ by $Z_{x,u}$; then $(Z_x)_u = Z_{x,u}$.

If (X, d) is a metric space, then X^n is assumed to be equipped with the sup metric $(a, b) \mapsto \sup\{\, \mathrm{d}(a_i, b_i) : i = 1, \dots, n \,\}$. In particular, we use the sup norm $|a| := \sup\{\, |a_i| : i = 1, \dots, n \,\}$ in $\mathbb{R}^n$.

For our purposes, **intervals** (in $\mathbb{R}$) always have interior. The usual notation is employed for the various kinds of intervals. We sometimes write $\mathbb{R}^{>0}$ or $(0, \infty)$ instead of $(0, +\infty)$. The interval $[0, 1]$ is denoted by $\mathbb{I}$. A **box** in $\mathbb{R}^n$ is an n-fold product of open intervals, and a **closed box** is a product of closed intervals.

The set E is **discrete** if all of its points are isolated, **locally closed** if it is open in its closure, and **dense in** $C \subseteq \mathbb{R}^n$ if $\mathrm{cl}(C \cap E) = \mathrm{cl}(C)$.

2.1. Some basic properties of ind in $\mathbb{R}^n$ ([9, 1.8]):
  (1) $\operatorname{ind} E = n$ if and only if E is nonempty and open.
  (2) ind is increasing.
  (3) ind is **countably stable** on closed sets, that is, $\operatorname{ind} \bigcup_{k \in \mathbb{N}} F_k = \max_{k \in \mathbb{N}} \operatorname{ind} F_k$ for any sequence $(F_k)_{k \in \mathbb{N}}$ of closed subsets of $\mathbb{R}^n$. (Hence, if E is F_σ, then $\operatorname{ind} E$ is realized on some compact subset of E.)
  (4) ind is **sublogarithmic**, that is, if $A \subseteq \mathbb{R}^j$ and $B \subseteq \mathbb{R}^k$, then $\operatorname{ind}(A \times B) \leq \operatorname{ind} A + \operatorname{ind} B$.
  (5) $\operatorname{ind} f(E) = \operatorname{ind} E$ for homeomorphisms $f \colon E \to f(E) \subseteq \mathbb{R}^p$ and $p \in \mathbb{N}$ (**topological invariance**).

2.2. More on dim in $\mathbb{R}^n$ (proofs are exercises):
  (1) dim is countably stable on closed sets (by the Baire Category Theorem).
  (2) For $k \leq m \leq n$ and $*$ in $\{\leq, \geq, =, <, >\}$, the set $\{\, x \in \mathbb{R}^m : \dim E_x * k \,\}$ is definable.

Note to logicians: (3) holds with "$\emptyset$-definable in $(\mathbb{R}, <, E)$" instead of "definable in $\mathfrak{R}$".

2.3. Some relations between dim and ind in $\mathbb{R}^n$ (proofs are exercises):
  (1) $\operatorname{ind} = \dim$ on subsets of $\mathbb{R}$.
  (2) $\dim E = 0 \Rightarrow \operatorname{ind} E = 0$.
  (3) $\operatorname{ind} \leq \dim$ on subsets of $\mathbb{R}^2$.
  (4) $\operatorname{ind} \leq \dim$ on compact sets [9, Exercise 1.8.C].
  (5) $\operatorname{ind} \leq \dim$ on F_σ sets.
  (6) $\operatorname{indcl} \leq \operatorname{dimcl}$.

(For (5), use (4) and that ind and dim are countably stable on closed sets.)

Remark. There are G_δ sets in $\mathbb{R}^3$ such that $\operatorname{ind} > \dim$; see [9, 1.10.23].

2.4. If C is the standard "middle-thirds" Cantor set, then $\operatorname{ind}\{\, (x - y, x + y) : x, y \in C \,\} = 0$ and $\dim\{\, (x - y, x + y) : x, y \in C \,\} = 1$ (recall that the difference set of C has interior).

Hence, strict inequality is possible in 2.3.3 even on compact sets. This also shows that dim is not necessarily preserved under isometries. (This elegant example was brought to our attention by A. Fornasiero.)

2.5. We shall have need later for notions of uniformity for collections of sets. Given a set X, we say that a map $f \colon \mathcal{P}(\mathcal{P}(X)) \to \mathbb{R} \cup \{\pm\infty\}$ is **increasing** if $f(\mathcal{A}) \leq f(\mathcal{B})$ for all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ such that $\mathcal{A}$ is a refinement of $\mathcal{B}$, and **stable** if

$$f(\{\, A \cup B : (A, B) \in \mathcal{A} \times \mathcal{B} \,\}) = \max(f(\mathcal{A}), f(\mathcal{B}))$$

for all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. There are obvious modifications of "logarithmic" and "sublogarithmic" for cartesian products. For illustrative purposes, if X is a topological space, then we put $\dim \mathcal{A} = \sup\{\, \dim A : A \in \mathcal{A} \,\}$ for $\mathcal{A} \subseteq \mathcal{P}(X^m)$, suppressing the subscript on $\dim_m$ as usual. Some easy basic facts:

— $\dim A = \dim\{A\}$ for all $A \subseteq X^m$.
— If $\mathcal{A} \subseteq \mathcal{P}(X^m)$, then $\dim \mathcal{A} = m$ if and only if some $A \in \mathcal{A}$ has interior.
— If $\mathcal{A} \subseteq \mathcal{P}(X)$, then $\dim \mathcal{A} = 0$ if and only if some $A \in \mathcal{A}$ is nonempty and no $A \in \mathcal{A}$ has interior.
— dim is increasing.
— dim is stable on the collection of compact subsets of X^m.
— dim is logarithmic.

The **open core** of $\mathfrak{R}$, denoted by $\mathfrak{R}^\circ$, is the expansion of the set $\mathbb{R}$ by all open sets (of any arity) definable in $\mathfrak{R}$. (Equivalently: $\mathfrak{R}^\circ$ is identified with the smallest structure $\mathfrak{S}$ on $\mathbb{R}$ such that each $\mathfrak{S}_m$ contains every open subset of $\mathbb{R}^m$ definable in $\mathfrak{R}$.) Some easy observations:

2.6.   *(1) $\mathfrak{R}^\circ$ is interdefinable with the expansion of $\overline{\mathbb{R}}$ by all projections of closed sets definable in $\mathfrak{R}$.*

   (2) $\mathfrak{R}$ defines $\mathbb{N}$ if and only if $\mathfrak{R}^\circ$ defines $\mathbb{N}$ if and only $\mathfrak{R}^\circ$ is interdefinable with $(\overline{\mathbb{R}}, \mathbb{N})$.

The expansion $\mathfrak{R}$ is **o-minimal** if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals. O-minimal expansions of $\overline{\mathbb{R}}$ are often regarded as the best-behaved expansions of $\overline{\mathbb{R}}$, indeed, they have so many nice properties that it takes several pages just to state even the most important ones; see, *e.g.*, [8, §4]. It is immediate from definitions that

2.7. *$\mathfrak{R}$ is o-minimal if and only if $\mathfrak{R}^\circ$ is o-minimal and every subset of $\mathbb{R}$ definable in $\mathfrak{R}$ has interior or is nowhere dense.*

More substantial:

2.8. *$\mathfrak{R}^\circ$ is o-minimal if and only if $\mathfrak{R}$ defines no infinite discrete closed subsets of $\mathbb{R}$.*

Proof. The forward implication is immediate from definitions. Suppose that every discrete closed subset of $\mathbb{R}$ definable in $\mathfrak{R}$ is finite. By [19, Lemma 2] every discrete subset of $\mathbb{R}$ definable in $\mathfrak{R}$ is finite. By Miller and Speissegger [27, Theorem (b)], $\mathfrak{R}^\circ$ is o-minimal. $\square$

For some examples of non-o-minimal expansions of $\overline{\mathbb{R}}$ having o-minimal open core, see: Belegradek and Zilber [1]; Dolich, Miller and Steinhorn [3, 1.11 and 1.12] and [4]; van den Dries [7, Theorem 5]; Günaydın and Hieronymi [18]; and [27, 4.2].

Projections of closed definable sets. Following [27], we let $\mathrm{D}_{\Sigma}(n)$ denote the collection of all subsets of $\mathbb{R}^n$ that are projections of closed definable sets. If the ambient $\mathbb{R}^n$ is irrelevant or understood from context, then we usually write just D_{Σ}. (Thus, we can now state the conclusion of Theorem A as "ind = Dim = dim on D_{Σ}".) In this subsection, we review some basic known facts and establish a key technical result (2.15) that disposes of the "ind = dim" part of Theorem A.

Remark. As every D_{Σ} set is F_{σ}, one might be tempted to think of "D_{Σ}" as abbreviating "definably F_{σ}", but caution is in order, as there at least two other reasonable notions for this: (i) definable and F_{σ}; (ii) definable and a countable union of closed definable sets. In general, these notions differ from D_{Σ}, as well as from each other.

2.9. (1) D_{Σ} *is closed under projection, cartesian product, union and intersection.*
(2) *Every constructible definable set is D_{Σ}.*
(3) *If $Z \in \mathrm{D}_{\Sigma}(m+n)$ and $x \in \mathbb{R}^m$, then $Z_x \in \mathrm{D}_{\Sigma}(n)$.*

Proof. (1). Closure under projection is immediate from definition, and closure under cartesian product is nearly so. We now deal with unions and intersections. Let $E_1, E_2 \in \mathrm{D}_{\Sigma}(n)$. For $i = 1, 2$, let $F_i \subseteq \mathbb{R}^{p_i}$ be closed and definable such that E_i is a projection of F_i. By permutation of coordinates, we may assume that E_i is the projection of F_i on the first n coordinates. After embedding into a common $\mathbb{R}^p$, we may take $p_1 = p_2 = p$. Then $E_1 \cup E_2$ is the projection on the first n coordinates of $F_1 \cup F_2$, and $E_1 \cap E_2$ is the projection on the first n coordinates of

$$\{ (x, y, z) \in \mathbb{R}^{n+2p} : (x, y) \in F_1 \ \& \ (x, z) \in F_2 \}.$$

(2). By Dougherty and Miller [5], every constructible definable set is a boolean combination of open definable sets. Hence, as D_{Σ} contains all closed definable sets and is closed under unions and intersections, it suffices to show that all open definable sets are D_{Σ}. If E is open (or even just locally closed), then it is the projection on the first n coordinates of

$$\{ (x, t) \in E \times \mathbb{R} : t \, \mathrm{dist}(x, (\mathrm{cl}\, E) \setminus E) \geq 1 \}. \qquad \square$$

(3). Z_x is the projection of $Z \cap (\{x\} \times \mathbb{R}^n)$ on the last n coordinates.

D_{Σ} need not be closed under complementation; indeed, complements of D_{Σ} sets need not be F_{σ} (*e.g.*, if $\mathfrak{R}$ defines $\mathbb{N}$, then $\mathbb{Q}$ is D_{Σ} and $\mathbb{R} \setminus \mathbb{Q}$ is not F_{σ}). If D_{Σ} is closed under complementation, then $\mathrm{D}_{\Sigma} = \mathfrak{S}(\mathfrak{R}^\circ)$. An aside to logicians:

2.10. E *is D_{Σ} if and only if E is existentially definable in $\mathfrak{R}^\circ$. The following are equivalent.*

— D_{Σ} *is closed under complementation.*
— $\mathrm{Th}(\mathfrak{R}^\circ)$ *is model complete.*
— *Every set definable in $\mathfrak{R}^\circ$ is D_{Σ}.*

We use the next result often.

2.11. *The following are equivalent.*

(1) E *is D_{Σ}.*
(2) *There is a definable $X \subseteq \mathbb{R}^{>0} \times \mathbb{R}^n$ such that $E = \bigcup_{r>0} X_r$ and $(X_r)_{r>0}$ is an increasing family of compact sets.*
(3) E *is the projection on the first n coordinates of a closed definable $F \subseteq \mathbb{R}^{n+1}$.*

(4) There are a closed definable $F \subseteq \mathbb{R}^{n+1}$ and a surjective continuous definable map $f\colon F \to E$.

(5) There are a closed definable $F \subseteq \mathbb{R}^p$ and a surjective continuous definable map $f\colon F \to E$.

Proof. For (1)$\Rightarrow$(2), if E is the projection πF of a closed definable $F \subseteq \mathbb{R}^p$, then put $X = \{\, (r,x) : x \in \pi(F \cap [-r,r]^p) \,\}$. For (2)$\Rightarrow$(3), let F be the closure of $\{\, (x,r) \in \mathbb{R}^{n+1} : x \in X_r \,\}$. The rest is routine. $\square$

2.12. *If E is D_Σ, then $\{\, x \in \mathbb{R}^m : \dim E_x \geq k \,\}$ is D_Σ.*

Proof. As D_Σ is closed under finite unions and permutation of coordinates, it suffices to let π be a projection $\mathbb{R}^{n-m} \to \mathbb{R}^k$ and show that $S := \{\, x \in \mathbb{R}^m : \pi(E_x) \text{ has interior } \}$ is D_Σ. Write $E = \bigcup_{r>0} X_r$ as in 2.11; then

$$S = \bigcup_{r>0} \{\, x \in \mathbb{R}^m : \exists y \in \mathbb{R}^k \; \textstyle\prod_{i=1}^{k} [y_i, (1+1/r)y_i] \subseteq \pi(X_{r,x}) \,\}.$$

Let $x \in \mathbb{R}^m$ and $r > 0$. As X_r is compact, so is $\pi(X_{r,x})$, hence so is

$$\{\, x \in \mathbb{R}^m : \exists y \in \mathbb{R}^k \; \textstyle\prod_{i=1}^{k} [y_i, (1+1/r)y_i] \subseteq \pi(X_{r,x}) \,\}.$$

The result follows. $\square$

We say that a map f is D_Σ if it is D_Σ as a set (that is, if its graph is D_Σ).

2.13. *If $f\colon E \to \mathbb{R}^p$ is D_Σ, then so are E, $f(E)$, and $f^{-1}(Y)$ for every $Y \in D_\Sigma(p)$.*

Proof. Observe that E is the projection of f on the first n coordinates, $f(E)$ is the projection of f on the last p coordinates, and $f^{-1}(Y)$ is the projection on the first n coordinates of $f \cap (\mathbb{R}^n \times Y)$. $\square$

Given $f\colon E \to \mathbb{R}^p$, we let $\mathcal{D}(f)$ denote the set of all $x \in E$ such that f is discontinuous at x. For $s > 0$, we let $\mathcal{D}(f,s)$ be the set of $x \in E$ such that the oscillation of f at x (defined as is usual in analysis) is at least s. Observe that $\mathcal{D}(f,s)$ is closed in E and $\mathcal{D}(f) = \bigcup_{s>0} \mathcal{D}(f,s)$. If f is definable, then so is $\mathcal{D}(f,s)$.

2.14. *If E is D_Σ and $f\colon E \to \mathbb{R}^p$ is definable, then $\mathcal{D}(f)$ is D_Σ.*

Proof. If $E = \bigcup_{r>0} X_r$ as in 2.11, then $\mathcal{D}(f) = \bigcup_{r>0} \mathcal{D}(f{\restriction}X_r, 1/r)$. $\square$

Recall that a closed box in $\mathbb{R}^n$ is the closure of a nonempty open box in $\mathbb{R}^n$. We define **rectangular partitions** of closed boxes by induction: A rectangular partition of a closed interval $B \subseteq \mathbb{R}$ is a finite cover of B by closed subintervals whose interiors are pairwise disjoint; a rectangular partition of a closed box $B \subseteq \mathbb{R}^{n+1}$ is the cartesian product of a rectangular partition of the projection of B on the first n variables with a rectangular partition of the projection of B on the last variable.

Next is a key technical result.

2.15. Proposition. *The following are equivalent.*

(1) $\mathfrak{R}$ *does not define* $\mathbb{N}$.

(2) *For all definable $h\colon A^p \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ is discrete, $h(A^p)$ is nowhere dense.*

(3) *If E is D_Σ, then for every $m \le n$ and closed box $B \subseteq \mathbb{R}^{n-m}$ there is a rectangular partition $\mathcal{P}$ of B such that for each $u \in \mathbb{R}^m$ there exists $P \in \mathcal{P}$ disjoint from $E_u \setminus \mathrm{int}(E_u)$.*

(4) *Each D_Σ either has interior or is nowhere dense.*

(5) $\mathrm{ind} = \dim$ *on* D_Σ.

(6) $\mathrm{ind} \ge \dim$ *on all compact definable subsets of* $\mathbb{R}^2$.

(The implication (1)$\Rightarrow$(3) can be regarded as a uniform version of (1)$\Rightarrow$(2), which is a theorem on its own; see [19].)

Proof. (1)$\Rightarrow$(2) is a special case of [19, Theorem A].

(2)$\Rightarrow$(3). By replacing E with $\mathbb{R} \times E$, we may assume that $m > 0$. The result is trivial if $m = n$, and an easy induction allows us to reduce to the case that $n = m + 1$, that is, each $E_u \subseteq \mathbb{R}$.

We first do the case that E is bounded. Since D_Σ is closed under intersection and continuous definable maps, we reduce to the case that $E \subseteq \mathbb{I}^{m+1}$ and $B = \mathbb{I}$ (recall that $\mathbb{I} = [0, 1]$). It suffices now to find $\epsilon > 0$ such that if $u \in \mathbb{I}^m$, then at least one of E_u or $\mathbb{I} \setminus E_u$ contains an interval of length ϵ. This follows easily from cell decomposition (see, e.g., [8, 4.2]) if $\mathfrak{R}^\circ$ is o-minimal, so assume that $\mathfrak{R}^\circ$ is not o-minimal. By 2.8, $\mathfrak{R}$ defines an infinite discrete closed $D \subseteq \mathbb{R}^{>0}$, and we have $E = \bigcup_{r \in D} X_r$ where $(X_r)_{r \in D}$ is an increasing definable family of compact sets. Define $Y, Z \subseteq D^2 \times \mathbb{I}^m$ by $u \in Y_{r,s}$ if $X_{r,u}$ contains an interval of length $1/s$, and $u \in Z_{r,s}$ if every interval contained in $\mathbb{I} \setminus X_{r,u}$ has length at most $1/s$. It suffices to show that there is some $r \in D$ such that $Z_{r,s} \setminus Y_{r,s} = \emptyset$ for every $s \in D \cap (r, \infty)$; suppose this fails. Then we have a definable function $\alpha \colon D \to D$ given by

$$\alpha(r) = \min\{\, s \in D \cap (r, \infty) : Z_{r,s} \setminus Y_{r,s} \ne \emptyset \,\}.$$

Observe that for every $r \in D$, both $Y_{r,\alpha(r)}$ and $Z_{r,\alpha(r)}$ are compact (for the former, recall the proof of 2.12). Let $r \in D$. If $Z_{r,\alpha(r)} \cap Y_{r,\alpha(r)} = \emptyset$, let $\beta(r)$ be the lexicographic minimum of $Z_{r,\alpha(r)}$. If $Z_{r,\alpha(r)} \cap Y_{r,\alpha(r)} \ne \emptyset$, let $\beta(r)$ be the lexicographic minimum of all $u \in Z_{r,\alpha(r)}$ such that the distance of u to $Y_{r,\alpha(r)}$ is maximal. Then $\beta \colon D \to \mathbb{R}^m$ is definable and $\beta(r) \in Z_{r,\alpha(r)} \setminus Y_{r,\alpha(r)}$ for all $r \in D$. Since X_r is closed, so is $X_{r,u} \setminus \mathrm{int}(X_{r,u})$. Define $M \subseteq D \times \mathbb{I}^m \times \mathbb{I}$ by $(r, u, t) \in M$ if and only t is a midpoint of a complementary-in-$\mathbb{I}$ interval of $X_{r,u} \setminus \mathrm{int}(X_{r,u})$. Let $\sigma(r)$ be the successor of r in D, that is, $\sigma(r) = \min(D \cap (r, \infty))$. Now M is definable, each $M_{r,u}$ is discrete, and the image of $M_{\alpha(r),\beta(r)}$ under the function $x \mapsto r + (\sigma(r) - r)x \colon \mathbb{R} \to \mathbb{R}$ is discrete and contained in the interval $(r, \sigma(r))$. Let A be the union of all such sets as r ranges over D; then A is discrete and definable, and for every $a \in A$ there are unique $r \in D$ and $t \in M_{\alpha(r),\beta(r)}$ such that $a = r + (\sigma(r) - r)t$. Let $g \colon M \to \mathbb{R}$ be given by $(r, u, t) \mapsto \sup((X_{r,u} \setminus \mathrm{int}(X_{r,u})) \cap (-\infty, t])$, that is, $g(r, u, t)$ is the left endpoint of the complementary-in-$\mathbb{I}$ interval of $X_{r,u} \setminus \mathrm{int}(X_{r,u})$ that contains t. Then g is definable and, for each (r, u), the image $g(r, u, M_{r,u})$ is dense in $X_{r,u} \setminus \mathrm{int}(X_{r,u})$. Let $h \colon A \to \mathbb{R}$ send $r + (\sigma(r) - r)t$ in A to $g(\alpha(r), \beta(r), t)$; then h is definable and $h(A)$ is dense in $\mathbb{I}$, contradicting (2). (This finishes the case that E is bounded.)

If E is not bounded, apply the result to the image of E under the map

$$x \mapsto \left(x_1(1 + x_1^2)^{-1/2}, \ldots, x_n(1 + x_n^2)^{-1/2} \right)$$

and then pull back.

For (3)$\Rightarrow$(4), set $m = 0$.

(4)⇒(5). Suppose that every D_Σ has interior or is nowhere dense. Let E be D_Σ; we must show that $\dim E = \operatorname{ind} E$. By countable stability, it suffices to consider the case that E is compact. Since $\operatorname{ind} \leq \dim$ on compact sets, it suffices to show that $\operatorname{ind} E \geq \dim E$. By permuting coordinates, we reduce to the case that the projection of E on the first $\dim E$ coordinates contains a box V. Define $f\colon V \to \mathbb{R}^{n-\dim E}$ by letting $f(v)$ be the lexicographic minimum of E_v. By basic real analysis (or *cf.* [2, 2.8]), $\mathcal{D}(f)$ has no interior; by 2.14 and (4), it is nowhere dense. Thus, there is a box $U \subseteq V$ such that $f{\upharpoonright}U$ is continuous, and so $u \mapsto (u, f(u))$ maps U homeomorphically onto the graph of $f{\upharpoonright}U$. Hence, $\operatorname{ind} E \geq \operatorname{ind}(f{\upharpoonright}U) = \dim E$.

For (6)⇒(1), recall 2.4 and that $(\overline{\mathbb{R}}, \mathbb{N})$ defines all Borel sets. □

3. Prelude to proof of Theorem A

We establish in this section some basic cases of Theorem A whose proofs use at most two more properties of Dim in addition to those listed in 1.2. We do this for two reasons: (i) certain key ideas are well illustrated in this simple setting; (ii) there may be other notions of metric dimension beyond Dim amenable to the same proofs. Recall that by 2.15 we have already disposed of the ind = dim part of Theorem A.

3.1. *If $\mathfrak{R}^\circ$ is o-minimal, then* $\dim = \operatorname{Dim}$ *on* D_Σ.

Proof. Since every D_Σ is definable in $\mathfrak{R}^\circ$, it suffices to show that if $\mathfrak{R}$ is o-minimal and E is definable, then $\operatorname{Dim} E = \dim E$. By Kurdyka and Parusiński [22] (or see Fischer [11]), there exist $N \in \mathbb{N}$ and bi-Lipshitz maps $f_i\colon U_i \to \mathbb{R}^n$ for $i = 1, \ldots, N$ such that E is the union of the $f_i(U_i)$, each U_i is open in $\mathbb{R}^{n(i)}$, and $n_1 \leq \cdots \leq n_N = \dim E$. By Lipschitz invariance, $\operatorname{Dim} f_i(U_i) = \operatorname{Dim} U_i = n(i)$; by stability, $\operatorname{Dim} E = \max\{\, n_i : i = 1, \ldots, N \,\} = n_N = \dim E$. □

Remark. The proof suggests that if $\mathfrak{R}$ is o-minimal then any reasonable notion of metric dimension would always coincide with dim (hence also ind) on all definable sets.

Next we dispose of the $\dim 0$ case.

3.2. *If $E \in D_\Sigma$ and $0 = \dim E < \operatorname{Dim} E$, then $\mathfrak{R}$ defines* $\mathbb{N}$.

We require for the proof two more properties of Dim in addition to the first five listed in 1.2:

3.3. There exists $s > 1$ such that $\operatorname{Dim} A^2 \geq s \operatorname{Dim} A$ for all definable $A \subseteq \mathbb{R}$.

3.4. If $A \subseteq \mathbb{R}$ is definable and $\operatorname{Dim} A = 0$, then $\operatorname{Dim} A^k = 0$.

(Indeed, $\operatorname{Dim} E^k = k \operatorname{Dim} E$ always, but we do not yet need this.)

3.5. *If $A \subseteq \mathbb{R}$ is definable and $\operatorname{Dim} A > 0$, then there exist $N \in \mathbb{N}$ and $F\colon \mathbb{R}^N \to \mathbb{R}$ definable in $\overline{\overline{\mathbb{R}}}$ such that $F(A^N)$ is dense.*

(Definability of A is superfluous if one is interested only in Assouad dimension; see Remark (ii) below after the proof.)

Proof. Let $Q\colon \mathbb{R}^4 \to \mathbb{R}$ be given by

$$Q(x) = \begin{cases} (x_1 - x_2)/(x_3 - x_4), & x_3 \neq x_4 \\ 0, & x_3 = x_4 \end{cases}$$

We are done if $Q(A^4)$ is dense, so assume otherwise. Then the vector-difference set of A^2 is disjoint from some open double cone $C \subseteq \mathbb{R}^2$ centered at the origin. Let ℓ be the line through the origin perpendicular to the axis of C and let π be the orthogonal projection of $\mathbb{R}^2$ onto ℓ. The restriction of π to A^2 is bi-Lipschitz, and so $\operatorname{Dim} \pi(A^2) = \operatorname{Dim} A^2 \geq s \operatorname{Dim} A$, where s is as in 3.3. Let θ be the rotation of the plane taking ℓ to the real line; then $\operatorname{Dim}(\theta \circ \pi(A^2)) \geq s \operatorname{Dim} A$. We now have a linear function $T_1 \colon \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{Dim} T_1(A^2) \geq s \operatorname{Dim} A$. We are done if $Q([T_1(A^2)]^4)$ is dense, so assume otherwise, and repeat the process above to obtain a linear function $T_2 \colon \mathbb{R}^2 \to \mathbb{R}$ that maps $[T_1(A^2)]^2$ to a set having Dim at least $s^2 \operatorname{Dim} A$. Since $0 < \operatorname{Dim} A \leq 1 < s$, we obtain after finitely many repetitions of this procedure some $m \in \mathbb{N}$ and linear $T \colon \mathbb{R}^m \to \mathbb{R}$ such that $Q(T(A^m)^4)$ is dense, thus producing F as desired. $\qquad\square$

Remarks. (i) Up to the choice of the various finitely many cones C (equivalently, the rotations θ) the construction of F depends only on the least $k \in \mathbb{N}$ such that $s^k \operatorname{Dim} A > 1$. (ii) As mentioned earlier, $\operatorname{Dim} E^k = k \operatorname{Dim} E$ always. Hence, we could apply 3.5 to $\mathfrak{R} = (\overline{\mathbb{R}}, A)$ with $s = 2$, and so the assumption of definability is not essential for the conclusion.

Proof of 3.2. Let $E \in \mathrm{D}_\Sigma$ and $0 = \dim E < \operatorname{Dim} E$. We must show that $\mathfrak{R}$ defines $\mathbb{N}$.

First, suppose that $E \subseteq \mathbb{R}$. If E is somewhere dense, then $\mathbb{N}$ is definable by 2.15, so assume that E is nowhere dense. Then $\dim \operatorname{cl} E = 0$, so it suffices to consider the case that E is closed (since $\operatorname{Dim} = \operatorname{Dim} \operatorname{cl}$). Let D be the set of midpoints of the bounded complementary intervals of E, together with $1 + \max E$ if E is bounded above. Then D is discrete and the function $g \colon D \to \mathbb{R}$ mapping $t \in D$ to the left endpoint of the complementary interval of E that contains t is definable. As E is closed and has no interior, $g(D)$ is dense in E, so $\operatorname{Dim} g(D) > 0$ (because $\operatorname{Dim} \operatorname{cl} = \operatorname{Dim}$). With F as in 3.5 applied to $A = g(D)$, we have that $F(g(D)^N)$ is somewhere dense. Hence, there exists $m \in \mathbb{N}$ and definable $h \colon D^m \to \mathbb{R}$ such that $h(D^m)$ is somewhere dense. By 2.15, $\mathfrak{R}$ defines $\mathbb{N}$.

We now dispose of the general case. For $i = 1, \ldots, n$, let $\pi_i E$ be the projection of E on the i-th coordinate; then $\pi_i E \in \mathrm{D}_\Sigma(1)$ and $\dim \pi_i E = 0$. As $E \subseteq (\bigcup_{i=1}^n \pi_i E)^n$, we have $\operatorname{Dim} E \leq n \max(\operatorname{Dim} \pi_i E)$ by 3.4 and stability. Hence, $\operatorname{Dim} \pi_i E > 0$ for some i, and so $\mathfrak{R}$ defines $\mathbb{N}$ by the previous paragraph. $\qquad\square$

As an immediate corollary of 3.2,

3.6. *The following are equivalent.*

— $\mathfrak{R}$ *does not define* $\mathbb{N}$.
— $\dim = \operatorname{Dim}$ *on all* $\dim 0$ D_Σ *sets.*
— $\dim = \operatorname{Dim}$ *on all* D_Σ *sets in* $\mathbb{R}$.

(Recall that $\operatorname{Dim}\{\, 1/k : 0 \neq k \in \mathbb{N} \,\} \neq 0$.)

This is about as far as we can get by using only easily-stated basic facts like 3.3, 3.4 and those of 1.2, but a bit more discussion will help motivate some of the technicalities coming in the next section. As we shall see, $\operatorname{Dim}(A \times B) \leq \operatorname{Dim} A + \operatorname{Dim} B$ for all $A \in \mathbb{R}^j$ and $B \in \mathbb{R}^k$. Hence, if the projection πE of E on the first m coordinates has interior and the projection τE of E on the last $n - m$ coordinates has $\operatorname{Dim} 0$, then $\operatorname{Dim} E = \dim E$. But the case $\operatorname{Dim} \tau E = 0$ will be exceptional, so what can be said in general? Assume that E is D_Σ and $\mathfrak{R}$ does not define $\mathbb{N}$. It is easy to see via the Baire Category Theorem that the set $\{\, x \in \mathbb{R}^m : \dim E_x > 0 \,\}$ is meager; as it also D_Σ (2.12), it is nowhere dense by 2.15. Then

14

$\{\, x \in \mathbb{R}^m : \dim E_x = 0 \,\}$ has interior, so by monotonicity, stability and 3.2, we may reduce to the case that $\mathrm{Dim}\, E_x \leq 0$ for all $x \in \mathbb{R}^m$. One can now imagine that under appropriate inductive assumptions we should be able to reduce to the case that $n = m + 1$, and even that for every permutation σ of coordinates, $\pi\sigma E$ has interior and $\mathrm{Dim}((\sigma E)_x) \leq 0$ for every $x \in \mathbb{R}^m$. But here our naive axiomatic approach seems to stall, even for $n = 2$. (The reader who wishes to get an idea of the difficulties should consider trying to show now that if $g \colon [0,1] \to \mathbb{R}$ is continuous and $(\overline{\mathbb{R}}, g)$ does not define $\mathbb{N}$, then $\mathrm{Dim}\, g = 1$.)

4. Dim, $\overline{\dim}_{\mathrm{M}}$, AND THE PROOF OF THEOREM A

Given a metric space (X, d), $S \subseteq X$ and $r > 0$ put

$$\mathrm{net}_r S = \sup_{k \in \mathbb{N}} \exists x_1 \ldots x_k \in S, \ \bigwedge_{i \neq j} \mathrm{d}(x_i, x_j) \geq r.$$

The **Assouad dimension** of S, denoted in this paper by $\mathrm{Dim}\, S$, is the infimum of $\alpha \in \mathbb{R}$ such that the set

$$\{\, (r/R)^\alpha \, \mathrm{net}_r\{\, y \in S : \mathrm{d}(y, x) \leq R \,\} : x \in S, \ 0 < r < R < +\infty \,\}$$

is bounded. Our primary source for information on Dim is [23], but there: (a) Dim is denoted by $\dim_{\mathrm{A}}$, ind by dim, and our notion of dim is not used in any explicit sense; (b) the infimum is taken over $\alpha \geq 0$ (and so $\dim_{\mathrm{A}} \emptyset = 0$, as opposed to $\mathrm{Dim}\, \emptyset = -\infty$ by our definition). In addition to the properties listed in 1.2, Dim is sublogarithmic and logarithmic on cartesian powers.

Recall that by 2.15 our goal is to show that $\mathrm{Dim} = \dim$ on D_Σ if $\mathfrak{R}$ does not define $\mathbb{N}$. It turns out to be convenient to work with another notion of dimension for collections of sets. We define the **upper Minkowski dimension** of $\mathcal{S} \subseteq \mathcal{P}(X)$ by

$$\overline{\dim}_{\mathrm{M}} \mathcal{S} = \varlimsup_{\substack{r \downarrow 0 \\ S \in \mathcal{S}}} \frac{\log \mathrm{net}_r S}{-\log r},$$

(with $\log(0) = -\infty$ and $\log(+\infty) = +\infty$), that is, $\overline{\dim}_{\mathrm{M}} \mathcal{S}$ is the supremum of all $\alpha \in [-\infty, +\infty]$ for which there is a sequence (r_k) of positive real numbers and a sequence (S_k) of elements of $\mathcal{S}$ such that $\lim_{k \to +\infty} r_k = 0$ and $\lim_{k \to +\infty} (\log \mathrm{net}_{r_k} S_k)/(-\log r_k) = \alpha$. For $S \subseteq X$, we write $\overline{\dim}_{\mathrm{M}} S$ instead of $\overline{\dim}_{\mathrm{M}}\{S\}$. There are many different names for, and equivalent formulations of, $\overline{\dim}_{\mathrm{M}}$ on sets in the literature; it is often defined only for nonempty totally bounded sets S. (In [23], $\overline{\dim}_{\mathrm{M}}$ on sets is denoted by $\overline{\dim}_{\mathrm{B}}$ and is called the upper box-counting box dimension.)

Note. Equivalent metrics yield the same $\overline{\dim}_{\mathrm{M}}$ (and Dim). Indeed, if $f \colon (0, \delta) \to \mathbb{R}$ and $0 < a < b \in \mathbb{R}$ are such that $af(r) \leq \mathrm{net}_r S \leq bf(r)$ for all $r \in (0, \delta)$, then $\overline{\dim}_{\mathrm{M}} S = \varlimsup_{r \downarrow 0} (\log f(r))/(-\log r)$. These observations allow for switching between alternate definitions of $\overline{\dim}_{\mathrm{M}}$ (and Dim) as convenient; see [10, 3.1] for more details.

We shall obtain Theorem A from the following more technical result.

Theorem B. *If $\mathfrak{R}$ does not define $\mathbb{N}$ and E is D_Σ and bounded, then*

$$\overline{\dim}_{\mathrm{M}}\{\, E_x : x \in \mathbb{R}^m \ \& \ \dim E_x = d \,\} \leq d, \quad d = 0, \ldots, n - m.$$

We begin to work toward the proof by listing some basic properties of $\overline{\dim}_{\mathrm{M}}$ in arbitrary metric spaces (proofs are easy and are left to the reader).

4.1. $\sup\{\,\overline{\dim}_{\mathrm{M}}S : S \in \mathcal{S}\,\} \leq \overline{\dim}_{\mathrm{M}}\mathcal{S} \leq \overline{\dim}_{\mathrm{M}}\bigcup\mathcal{S}$.

Recalling 2.5 for definitions, $\overline{\dim}_{\mathrm{M}}$ is increasing, stable, sublogarithmic, and logarithmic on cartesian *powers*. It is also closure invariant:

4.2. $\overline{\dim}_{\mathrm{M}}\operatorname{cl}\mathcal{S} = \overline{\dim}_{\mathrm{M}}\mathcal{S}$, *where* $\operatorname{cl}\mathcal{S} := \{\,\operatorname{cl}S : S \in \mathcal{S}\,\}$.

4.3. *If* $\mathcal{A} \subseteq \mathcal{P}(X^n)$, *then* $\dim\mathcal{A} \leq \overline{\dim}_{\mathrm{M}}\mathcal{A}$.

If Y is another metric space, then $\overline{\dim}_{\mathrm{M}}$ is "Lipschitz nonexpansive":

4.4. *If* $(f_S : S \to Y)_{S \in \mathcal{S}}$ *is uniformly Lipschitz, then* $\overline{\dim}_{\mathrm{M}}\{\,f_S(S) : S \in \mathcal{S}\,\} \leq \overline{\dim}_{\mathrm{M}}\mathcal{S}$.

Hence, $\overline{\dim}_{\mathrm{M}}$ is also **Lipschitz invariant**:

4.5. *If* $(f_S : S \to Y)_{S \in \mathcal{S}}$ *is uniformly bi-Lipschitz, then* $\overline{\dim}_{\mathrm{M}}\{\,f_S(S) : S \in \mathcal{S}\,\} = \overline{\dim}_{\mathrm{M}}\mathcal{S}$.

For $X = \mathbb{R}^n$, we have a special property.

4.6. $\overline{\dim}_{\mathrm{M}}\mathcal{A} \leq n$ *for every* $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}^n)$ *such that* $\sup\{\,\operatorname{diam}A : A \in \mathcal{A}\,\} < +\infty$.

When combined with 4.1,

4.7. *If* E *is bounded and some* E_x *has interior* $(x \in \mathbb{R}^m)$, *then* $\overline{\dim}_{\mathrm{M}}\{\,E_x : x \in \mathbb{R}^m\,\} = n-m$.

Hence,

4.8. *Theorem B holds for* $d = n - m$.

Next is a uniform version of 3.2.

4.9. *Theorem B holds for* $d = 0$.

Proof. Assume that $\mathfrak{R}$ does not define $\mathbb{N}$ and E is D_Σ and bounded. Let $A = \{\,x \in \mathbb{R}^m : \dim E_x = 0\,\}$. We must show that $\overline{\dim}_{\mathrm{M}}\{\,E_x : x \in A\,\} \leq 0$.

First, suppose that $n = m + 1$. By following the argument for 3.2, it suffices to exhibit a linear $T : \mathbb{R}^2 \to \mathbb{R}$ such that
$$\overline{\dim}_{\mathrm{M}}\{\,T((E_x)^2) : x \in A\,\} \geq 2\overline{\dim}_{\mathrm{M}}\{\,E_x : x \in A\,\}.$$
With Q as in the proof of 3.5, observe that $\{\,(x,t) : x \in \mathbb{R}^m\ \&\ t \in Q((E_x)^4)\,\}$ is D_Σ, and for every $x \in \mathbb{R}^m$, E_x has interior if and only if $Q((E_x)^4)$ has interior. Hence, by 2.15, there is a finite set $\mathcal{I}$ of intervals such that for every $x \in A$ there exists $I_x \in \mathcal{I}$ such that $Q((E_x)^4)$ is disjoint from I_x. By arguing as in 3.5 there is a finite collection $\mathcal{L}$ of lines through the origin and positive real numbers $(c_\ell)_{\ell \in \mathcal{L}}$ such that, for each $x \in A$, there exists $\ell(x) \in L$ such that the restriction to $(E_x)^2$ of the orthogonal projection $\pi_{\ell(x)}$ onto $\ell(x)$ is bi-Lipschitz with lower constant $c_{\ell(x)}$. For each such ℓ, let A_ℓ be the set of $x \in A$ such that $\pi_\ell{\restriction}(E_x)^2$ is bi-Lipschitz with lower constant c_ℓ. Let θ_ℓ be a rotation taking ℓ to the real line and put $T_\ell = \theta_\ell \circ \pi_\ell$. As the family $(T_\ell{\restriction}((E_x)^2))_{x \in A_\ell}$ is uniformly bi-Lipschitz, we have
$$\overline{\dim}_{\mathrm{M}}\{\,T_\ell((E_x)^2) : x \in A_\ell\,\} = \overline{\dim}_{\mathrm{M}}\{\,(E_x)^2 : x \in A_\ell\,\}.$$
By stability, for some $\ell \in \mathcal{L}$ we have
$$\overline{\dim}_{\mathrm{M}}\{\,(E_x)^2 : x \in A\,\} = \overline{\dim}_{\mathrm{M}}\{\,(E_x)^2 : x \in A_\ell\,\}.$$
Hence, for this ℓ, we have
$$\overline{\dim}_{\mathrm{M}}\{\,T_\ell((E_x)^2) : x \in A\,\} \geq \overline{\dim}_{\mathrm{M}}\{\,(E_x)^2 : x \in A\,\}.$$

16

As $\overline{\dim}_{\mathrm{M}}$ is logarithmic on cartesian powers, we have
$$\overline{\dim}_{\mathrm{M}}\{\, T_\ell((E_x)^2) : x \in A \,\} \geq 2\overline{\dim}_{\mathrm{M}}\{\, E_x : x \in \mathbb{R}^m \,\}$$
as desired.

With the case $n = m + 1$ in hand, the general case follows by the uniform version of the last paragraph of the proof of 3.2. $\qquad\square$

Hence,

4.10. *Theorem B holds if either $n - m = 1$ or $\dim\{\, E_x : x \in \mathbb{R}^m \,\} \leq 0$.*

Again we have made it this far in axiomatic fashion, that is, via easily-stated properties of $\overline{\dim}_{\mathrm{M}}$ rather than its definition; this now changes. We shall finish the proof of Theorem B by an induction, but we need to make the inductive assumptions rather more precise. We require some notation and definitions (some of which are formulated specifically for this induction).

Let $\Pi(k, j)$ denote the collection of all coordinate projections $\mathbb{R}^k \to \mathbb{R}^j$, and $\Pi(k) = \bigcup_j \Pi(k, j)$. Given $\pi \in \Pi(k, j)$ we let $\pi^\perp \in \Pi(k, k - j)$ denote the orthogonal projection. Observe that if $\pi \in \Pi(n, m)$ is the projection on the first m coordinates and $x \in \mathbb{R}^m$, then $E_x = \pi^\perp(E \cap \pi^{-1}(x))$.

Given a set A, its cardinality is denoted by $\operatorname{card} A$.

Put $B_k(E) = \{\, v \in \mathbb{Z}^n : (v + \mathbb{I}^n) \cap 2^k E \neq \emptyset \,\}$ and $N_k(E) = \operatorname{card} B_k(E)$. Note that $N_k(E)$ is the number (allowing $+\infty$) of closed dyadic cubes in $\mathbb{R}^n$ of side length 2^{-k} that intersect E, and that if $E \subseteq \mathbb{I}^n$, then $B_k(E) \subseteq \mathbb{N}^n$. If $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^n)$, then $B_k(\bigcup \mathcal{S}) = \bigcup_{S \in \mathcal{S}} B_k(S)$. It is a standard fact that $\overline{\dim}_{\mathrm{M}} E = \overline{\lim}_{k \to +\infty} \log_2 N_k(E)/k$. Hence,

4.11. *If $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^n)$, then $\overline{\dim}_{\mathrm{M}} \mathcal{S} \leq \alpha$ iff*
$$\forall \epsilon > 0 \exists k_\epsilon \in \mathbb{N} \forall k \geq k_\epsilon \forall S \in \mathcal{S}, \ N_k(S) \leq 2^{k(\alpha + \epsilon)}.$$

The next batch of definitions is for tracking certain data that we need to make our induction go through.

For $\delta \in (0, \infty)$, we say that $C \subseteq \mathbb{N}^n$ is **δ-sparse** if there exists $\pi \in \Pi(n, n - 1)$ such that $\operatorname{card}(\pi^\perp(C \cap \pi^{-1}(u))) \leq \delta$ for all $u \in \mathbb{N}^{n-1}$. (For $n = 1$, this means just that $\operatorname{card} C \leq \delta$.) If π is the projection on the first $n - 1$ coordinates, then this means that $\operatorname{card}(C_u) \leq \delta$ for all $u \in \mathbb{N}^{n-1}$. An easy observation:

4.12. *If $C \subseteq \mathbb{N}^n$ is δ-sparse, then there exists $\pi \in \Pi(n, n-1)$ such that $\operatorname{card} C \leq \delta \operatorname{card} \pi C$.*

We need a uniform version for collections of sets. We say that $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N}^n)$ is **δ-sparse** if there exists $\pi \in \Pi(n, n - 1)$ such that for all $u \in \mathbb{N}^{n-1}$ and all $C \in \mathcal{C}$, we have $\operatorname{card}(\pi^\perp(C \cap \pi^{-1}(u))) \leq \delta$. The quantifier complexity of the next step is sufficiently complicated that we express it in logical notation. We say that $\mathcal{S} \subseteq \mathcal{P}(\mathbb{I}^n)$ is **sparse** if:
$$\exists s \in \mathbb{N} \forall \epsilon > 0 \exists k_\epsilon \in \mathbb{N} \forall k > k_\epsilon \exists (C_{1,S})_{S \in \mathcal{S}} \ldots (C_{s,S})_{S \in \mathcal{S}} \forall S \in \mathcal{S}, \ B_k(S) = \bigcup_{i=1}^s C_{i,S}$$
where each family $(C_{i,S})_{S \in \mathcal{S}} \subseteq \mathcal{P}(\mathbb{N}^n)$ is $2^{k\epsilon}$-sparse. (We have suppressed any notation here for the dependence of the families $(C_{1,S})_{S \in \mathcal{S}}$ on k; we introduce it later as needed.) We say that s **witnesses** (or is a witness to) that $\mathcal{S}$ is sparse, and that E is sparse if $E \subseteq \mathbb{I}^n$ and $\{E\}$ is sparse. Some easy observations (proofs are left to the reader):

4.13. *Let $\mathcal{S} \subseteq \mathcal{P}(\mathbb{I}^n)$.*

(1) *If $\overline{\dim}_M \mathcal{S} = 0$, then $\mathcal{S}$ is sparse and witnessed by $s = 1$.*
(2) *If $\mathcal{S}$ is sparse with witness s and $\mathcal{S}' \subseteq \mathcal{P}(\mathbb{I}^n)$ is sparse with witness s', then $\{ S \cup S' : (S, S') \in \mathcal{S} \times \mathcal{S}' \}$ is sparse with witness $s + s'$.*
(3) *If there exists $\pi \in \Pi(n)$ such that $\pi\mathcal{S} := \{ \pi S : S \in \mathcal{S} \}$ is sparse, then $\mathcal{S}$ is sparse with the same witness.*
(4) *If there exists $\pi \in \Pi(n)$ such that $\overline{\dim}_M \pi\mathcal{S} = 0$, then $\mathcal{S}$ is sparse and witnessed by $s = 1$.*

4.14. *If E is the union of an increasing sequence $(S_r)_{r>0}$ of subsets of $\mathbb{I}^n$ and*
$$\{ S_{r,x} : r > 0, \ x \in \mathbb{I}^m \}$$
is sparse, then $\{ E_x : x \in \mathbb{I}^m \}$ is sparse with the same witness.

Proof. Let $s \in \mathbb{N}$ witness that $\{ S_{r,x} : r > 0, \ x \in \mathbb{I}^m \}$ is sparse. Let $\epsilon > 0$. We exhibit k_ϵ such that for all $k > k_\epsilon$ there are $2^{k\epsilon}$-sparse families $(D_{k,1,x})_{x \in \mathbb{I}_m}, \ldots, (D_{k,s,x})_{x \in \mathbb{I}^m}$ of subsets of $\mathbb{N}^{n-m}$ such that $B_k(E_x) = \bigcup_{i=1}^s D_{k,i,x}$ for every $x \in \mathbb{I}^m$. By assumption, there exists $k_\epsilon \in \mathbb{N}$ such that for all $k > k_\epsilon$ there are $2^{k\epsilon}$-sparse families
$$(C_{k,1,r,x})_{(r,x) \in \mathbb{R}^{>0} \times \mathbb{I}^m}, \ldots, (C_{k,s,r,x})_{(r,x) \in \mathbb{R}^{>0} \times \mathbb{I}^m}$$
of subsets of $\mathbb{N}^{n-m}$ such that $B_k(S_{r,x}) = \bigcup_{i=1}^s C_{k,i,r,x}$ for every $(r, x) \in \mathbb{R}^{>0} \times \mathbb{I}^m$. Let $k > k_\epsilon$ and $x \in \mathbb{I}^m$. Since $E_x \subseteq \mathbb{I}^{n-m}$, we have $B_k(E_x) \subseteq \{0, \ldots, 2^k\}^{n-m}$. Since (S_r) is increasing in r, so is $(S_{r,x})$. Thus, there exists $r(k, x) > 0$ such that $B_k(E_x) = B_k(S_{r(k,x),x})$. Hence,
$$B_k(E_x) = \bigcup_{i=1}^s C_{k,i,r(k,x),x}.$$

Since each $(C_{k,i,r,x})_{(r,x)}$ is $2^{k\epsilon}$-sparse, so is each $(C_{k,i,r(k,x),x})_{x \in \mathbb{I}^m}$. Put $D_{k,i,x} = C_{k,i,r(k,x),x}$. Then $(D_{k,1,x})_{x \in \mathbb{I}^m}, \ldots, (D_{k,s,x})_{x \in \mathbb{I}^m}$ are as desired. $\square$

Proof of Theorem B. Let E be D_Σ and bounded. We must show that
$$\overline{\dim}_M \{ E_x : x \in \mathbb{R}^m \ \& \ \dim E_x = d \} \leq d.$$
We have already established the cases $n - m = 1$, $n - m = d$, and $d = 0$. By translation, dilation and Lipschitz invariance, we may take $E \subseteq \mathbb{I}^n$. We now proceed by induction on $n - m \geq 2$ to establish the following:

(i)$_{n,m}$ $\{ E_x : x \in \mathbb{I}^m \ \& \ \text{int}(E_x) = \emptyset \}$ is sparse.
(ii)$_{n,m}$ $\overline{\dim}_M \{ E_x : x \in \mathbb{I}^m \ \& \ \dim E_x = d \} \leq d$, $\quad 1 \leq d \leq n - m - 1$.

Remark. Formally, we could start the induction with $n - m = 1$, but the key ideas will be better illustrated by doing the case $n - m = 2$ explicitly.

Proof of (i)$_{m+2,m}$. Suppose that $n = m + 2$. Let $A = \{ x \in \mathbb{I}^m : \text{int}(E_x) = \emptyset \}$. We must show that $\{ E_x : x \in A \}$ is sparse. Write $E = \bigcup_{r>0} X_r$ as in 2.11. By 4.14, it suffices to show that the collection $\{ X_{r,x} : r > 0, \ x \in A \}$ is sparse. In order to improve readability, we first suppress the parameters and show that (i)$_{2,0}$ holds, that is, if $E \subseteq \mathbb{I}^2$ ($\cong \mathbb{I} \times \mathbb{I}$) is definable, compact and has no interior, then E is sparse. The argument will be explicit enough to read off the needed uniformity in parameters.

18

Define $f\colon \mathbb{I}^2 \to \mathbb{R}$ by $f(x,y) = \min(E_x \cap [y,2])$ $(= \min((E \cup (\mathbb{I} \times \{2\})_x)))$. For $y \in \mathbb{I}$, let $f(\_,y)$ denote the function $x \to f(x,y)\colon \mathbb{I} \to \mathbb{R}$. Put

$$F = E \cup \{\, (x,y) \in \mathbb{I}^2 : x \in \mathcal{D}(f(\_,y)) \,\}.$$

For $y \in \mathbb{R}$, put $E^y = \{\, x \in \mathbb{R} : (x,y) \in E \,\}$ and $F^y = \{\, x \in \mathbb{R} : (x,y) \in F \,\}$. For each k, put

$$C_{k,1} = B_k(\{\, (x,y) \in \mathbb{I}^2 : \mathrm{int}(E_x) \neq \emptyset \,\})$$

$$C_{k,2} = B_k(\{\, (x,y) \in \mathbb{I}^2 : \mathrm{int}(E^y) \neq \emptyset \,\})$$

$$C_{k,3} = \{\, (u,v) \in B_k(E) : u \in B_k(F^{v/2^k}) \,\} \setminus C_{k,2}$$

$$C_{k,4} = B_k(E) \setminus (C_{k,1} \cup C_{k,2} \cup C_{k,3})$$

As $B_k(E)$ is the union of the $C_{k,i}$, it suffices to let $\epsilon > 0$ and show that, for each $i \in \{1,2,3,4\}$, $C_{k,i}$ is $2^{k\epsilon}$-sparse for all sufficiently large k.

We first dispose of the $C_{k,1}$ and $C_{k,2}$. As E is F_σ and has no interior, the same is true of $\{\, x \in \mathbb{I} : \mathrm{int}(E_x) \neq \emptyset \,\}$; as this set is D_Σ (2.12) and contained in $\mathbb{R}$, it has $\overline{\dim}_M$ at most 0 by 4.10. Thus, for sufficiently large k, the projection of $C_{k,1}$ on the first coordinate has cardinality at most $2^{k\epsilon}$; for such k, the set $C_{k,1}$ is $2^{k\epsilon}$-sparse (by definition). The $C_{k,2}$ are similarly handled via projection on the second coordinate.

Now we deal with the $C_{k,3}$. By compactness of $E \cup (\mathbb{I} \times \{2\})$, each $f(\_,y)$ is lower semicontinuous. It is routine real analysis that for each $\eta > 0$ the set

$$E \cup \{\, (x,y) \in \mathbb{I}^2 : x \in \mathcal{D}(f(\_,y),\eta) \,\}$$

is compact—and so F is D_Σ (recall the argument for 2.14)—and for each $y \in \mathbb{I}$, F^y has interior if and only if E^y has interior. If $y \in \mathbb{I}$ and E^y has no interior, then $F^y \setminus \mathrm{int}(F_y) = E^y \cup \mathcal{D}(f(\_,y))$, and so $\overline{\dim}_M\{\, F^y : y \in \mathbb{I} \ \& \ \mathrm{int}(E^y) = \emptyset \,\} = 0$ by applying 4.10 to $\{\, (y,x) : (x,y) \in F \,\}$. By 4.13, $\{\, F^y : y \in \mathbb{I} \ \& \ \mathrm{int}(E^y) = \emptyset \,\}$ is sparse (with witness 1). Hence, $C_{k,3}$ is $2^{k\epsilon}$-sparse for all sufficiently large k, because $\mathrm{card}\{\, u \in \mathbb{N} : (u,v) \in C_{k,3} \,\} \leq \mathrm{card}\, N_k(F^{v/2^k})$ for each $v \in \mathbb{N}$.

Finally, we deal with the $C_{k,4}$. By 4.9, we have $\overline{\dim}_M\{\, E_x : \mathrm{int}(E_x) = \emptyset \,\} = 0$. Hence, for all $x \in \mathbb{I}$ and sufficiently large k, if E_x has no interior, then $N_k(E_x) \leq 2^{k\epsilon}$; we show that $C_{k,4}$ is $2^{k\epsilon}$-sparse for such k by letting $(u,v) \in C_{k,4}$ and showing that $(C_{k,4})_u \subseteq B_k(E_{u/2^k})$. As $C_{k,4}$ is disjoint from $C_{k,3}$, the function

$$x \mapsto f(x, v/2^k) \colon [u/2^k, (u+1)/2^k] \to \mathbb{R}$$

is continuous. Moreover, its range is contained in $[v/2^k, (v+1)/2^k]$, for if not, then $u \in B_k(E^{(v+1)/2^k})$ by the Intermediate Value Theorem, contradicting the definition of $C_{k,4}$. Hence, $v \in B_k(E_{u/2^k})$ (as desired).

We have finished the proof of the case $m = 0$ and $n = 2$. For arbitrary m, recall the compact sets $X_{r,x}$ at the beginning of the proof. Put $f(r,x,a,b) = \min(X_{r,x,a} \cap [b,2])$ for $(a,b) \in \mathbb{I}^2$. Make the obvious modifications to the definitions of the $C_{i,k}$, and then proceed mutatis mutandis. (This ends the proof of (i)$_{m+2,m}$.)

Proof of (ii)$_{m+2,m}$. Suppose that $n = m + 2$. Put $A(1) = \{\, x \in \mathbb{R}^m : \dim E_x = 1 \,\}$. We must show that $\overline{\dim}_M\{\, E_x : x \in A(1) \,\} \leq 1$. Suppose to the contrary that there exist $\epsilon > 0$

and a sequence (x_k) in $A(1)$ such that

$$(1) \qquad\qquad \forall k, \; N_k(E_{x_k}) > 2^{k(1+\epsilon)}.$$

For each $\pi \in \Pi(2,1)$, we have $\overline{\dim}_M\{\, \pi(E_x) : x \in \mathbb{R}^m \,\} \le 1$. Hence, after throwing away finitely many x_k, we reduce to the case that also

$$(2) \qquad\qquad \forall k \forall \pi \in \Pi(2,1), \; N_k(\pi(E_{x_k})) < 2^{k(1+\epsilon/3)}.$$

By (i)$_{m+2,m}$, $\{\, E_x : x \in A(1) \,\}$ is sparse. Thus, there exist $s, j \in \mathbb{N}$ such that $B_j(E_{x_j})$ is a union of s-many $2^{j\epsilon/2}$-sparse sets and $2^{j(1+\epsilon/2)} \ge s2^{j(1+\epsilon/3)}$. By 4.12 and (1), there exists $\pi \in \Pi(2,1)$ such that $N_j(\pi(E_{x_j})) \ge N_j(E_{x_j})/s2^{j\epsilon/2} > 2^{j(1+\epsilon/3)}$, contradicting (2). (This ends the proof of (ii)$_{m+2,m}$.)

Inductive assumption. Let $n - m > 2$. Assume that (i)$_{n',m'}$ and (ii)$_{n',m'}$ hold for all $n', m' \in \mathbb{N}$ such that $0 \le n' - m' < n - m$.

Proof of (i)$_{n,m}$. This is quite similar to the proof of (i)$_{m+2,m}$, so we only hint at the needed modifications. As before, we reduce to the case that $m = 0$, that is, we show that if E is compact and has no interior, then E is sparse (and explicitly so). Define f as before, but with x ranging over $\mathbb{I}^{n-1}$, and define the associated set F in the obvious way. Let π denote projection on the first $n-1$ coordinates. Consider the following sets and collections, where E^y and F^y are defined similarly as before:

(1) $\{\, (x,y) \in E : x \notin \operatorname{int} \pi E \,\}$
(2) $\{\, (x,y) \in E : x \in \operatorname{int} \pi E \; \& \; \operatorname{int}(E_x) \ne \emptyset \,\}$
(3) $\{\, (x,y) \in E : \operatorname{int}(E^y) \ne \emptyset \,\}$
(4) $\{\, E_x : x \in \mathbb{I}^{n-1} \; \& \; \operatorname{int}(E_x) = \emptyset \,\}$
(5) $\{\, F^y : y \in \mathbb{I} \; \& \; \operatorname{int}(E^y) = \emptyset \,\}$

Since $\pi E \setminus \operatorname{int} \pi E$ is D_Σ and has no interior, it is sparse by (i)$_{n-1,0}$; then set (1) is also sparse, because

$$\pi\{\, (x,y) \in E : x \notin \operatorname{int} \pi E \,\} = \pi E \setminus \operatorname{int} \pi E.$$

Similarly, sets (2) and (3) are sparse because the sets

$$\{\, x \in \operatorname{int} \pi E : \operatorname{int}(E_x) \ne \emptyset \,\} \quad \{\, y \in \mathbb{I} : \operatorname{int}(E^y) \ne \emptyset \,\}$$

are D_Σ and have no interior. By (i)$_{n,n-1}$, collection (4) is sparse. By (i)$_{n,1}$ (applied to $\{\, (y,x) : (x,y) \in F \,\}$), collection (5) is sparse. Now proceed mutatis mutandis as for (i)$_{2,0}$.

Proof of (ii)$_{n,m}$. Let $d \in \{1, \ldots, n-m-1\}$ and $A(d) = \{\, x \in \mathbb{R}^m : \dim E_x = d \,\}$. We must show that $\overline{\dim}_M\{\, E_x : x \in A(d) \,\} \le d$. If $d = n - m - 1$, then the argument is essentially the same as for (ii)$_{m+2,m}$. If $d < n - m - 1$, then we apply (ii)$_{n-1,m}$ to πE for each $\pi \in \Pi(n-m, n-m-1)$, then finish as for (ii)$_{m+2,m}$ via the obvious modifications.

This ends the proof of Theorem B. $\qquad\qquad\square$

Proof of Theorem A. Let E be D_Σ. We must show that $\operatorname{ind} E = \dim E = \operatorname{Dim} E$. By 2.15, it suffices to show that $\operatorname{Dim} E = \dim E$. For $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, put

$$\phi(x,t) = (x_1(1+x_1^2)^{-1/2}, \ldots, x_n(1+x_n^2)^{-1/2}, t(1+t^2)^{-1/2}).$$

The set

$$F := \{\, (\phi(x,t), (y-x)/t) : t > 0; \; x, y \in E; \; |x - y| \le t \,\}$$

is D_Σ and bounded. For each $(x,t) \in E \times \mathbb{R}^{>0}$, we have $F_{\phi(x,t)} = [-1,1]^n \cap (E - \{x\})/t$. Thus, $\dim F_{\phi(x,t)} = \dim E$, so by Theorem B,

$$\dim E = \overline{\dim}_M\{\,[-1,1]^n \cap (E - \{x\})/t : (x,t) \in E \times \mathbb{R}^{>0}\,\}.$$

By translation invariance,

$$\dim E = \overline{\dim}_M\{\,(E \cap \bar{B}(x,t))/t : (x,t) \in E \times \mathbb{R}^{>0}\,\}$$

where $\bar{B}(x,t)$ is the closed cube of side length $2t$ centered at x. Let $\epsilon > 0$. We show that $\operatorname{Dim} E \leq \epsilon + \dim E$. There exists $\rho > 0$ such that for all $x \in E$, $t > 0$ and $0 < s < \rho$, we have $\operatorname{net}_s(E \cap \bar{B}(x,t))/t) \leq s^{-(\epsilon + \dim E)}$. Put $C = \max(1, \operatorname{net}_\rho \bar{B}(0,1))$. Let $x \in E$ and $0 < r < R$. If $r/R < \rho$, then

$$\operatorname{net}_r(E \cap \bar{B}(x,R)) = \operatorname{net}_{r/R}((E \cap \bar{B}(x,R))/R) \leq (R/r)^{\epsilon + \dim E} \leq C(R/r)^{\epsilon + \dim E}.$$

If $r/R \geq \rho$, then

$$\operatorname{net}_r(E \cap \bar{B}(x,R)) \leq \operatorname{net}_{\rho R}(E \cap \bar{B}(x,R))$$

$$\leq \operatorname{net}_{\rho R} \bar{B}(x,R) = \operatorname{net}_\rho \bar{B}(0,1) \leq C(R/r)^{\epsilon + \dim E}.$$

Hence, $\operatorname{Dim} E \leq \epsilon + \dim E$, as desired. $\qquad \square$

5. Concluding remarks

We have already noted in the introduction that Theorem A applies beyond the easy case that $\mathfrak{R}^\circ$ is o-minimal. Here is another class of examples not yet described in the literature.

5.1. *There is a Cantor set $E \subseteq \mathbb{R}$ such that, if $\mathfrak{R}$ is o-minimal and exponentially bounded, then $\operatorname{ind} = \operatorname{Dim} = \dim$ on all sets definable in the expansion of $\mathfrak{R}$ by all subsets of E^m (m ranging over $\mathbb{N}$) that have countable closure.*

(See any of [8, 15, 26] for the definition of "exponentially bounded". Note that every Cantor subset of $\mathbb{R}$ contains countable closed sets of arbitrary countable Cantor-Bendixson rank.)

Proof. Let E be as in the proof of [15, Theorem B]. By 1.8, it suffices to let $X \subseteq E$ be countable and closed, and show that every bounded unary set definable in $(\mathfrak{R}, X)^\#$ has interior or is countable, where $(\mathfrak{R}, X)^\#$ denotes the expansion of $\mathfrak{R}$ by all subsets of each finite cartesian power of X. By [15, 1.11], it suffices to let $f : [0,1]^n \to \mathbb{R}$ be bounded and definable in $\mathfrak{R}$, and show that $f(X^n)$ has countable closure. By compactness of X^n, it suffices to let $x \in X^n$ and find $\delta > 0$ such that $f(X^n \cap B(x,\delta))$ has countable closure. The result now follows from [15, 1.8], assertion (ii) in the proof of [15, Theorem B], and the countability and compactness of X^n. $\qquad \square$

5.2. Consideration of $\{\,k! : k \in \mathbb{N}\,\}$ shows that the dimensional coincidence $\operatorname{ind} = \dim = \operatorname{Dim}$ can hold for sets that define $\mathbb{N}$ over $\bar{\mathbb{R}}$.

5.3. In 1.7, "F_σ" cannot be relaxed to "boolean combination of F_σ": By [7, Theorem 1], if E is F_σ and a real-closed proper subfield of $\mathbb{R}$ (say, if E is the set of all real algebraic numbers), then every set definable in $(\bar{\mathbb{R}}, E)$ is a boolean combination of F_σ sets. But both E and $\mathbb{R} \setminus E$ are dense in $\mathbb{R}$.

5.4. The conclusion of Theorem A does not generally extend to definable metric spaces (that is, metric spaces of the form (E, d) where $d\colon E^2 \to \mathbb{R}^{\geq 0}$ is definable). For example, $(x, y) \mapsto |x - y|^{2/3}$ is a metric on $\mathbb{I}$, and it is easy to see that $\dim_{\mathrm{H}} \mathbb{I} = 3/2$ with respect to this metric.

Open issues and further directions. While we regard Theorem A as a major advance in the tameness program over $\overline{\mathbb{R}}$, there are several questions left open. Perhaps the most important of these: If $\mathfrak{R}$ does not define $\mathbb{N}$, does $\mathrm{ind} = \mathrm{Dim} = \dim$ on all sets definable in the open core of $\mathfrak{R}$? To put this another way, if $\mathfrak{R}$ is an expansion of $\overline{\mathbb{R}}$ by constructible sets and it does not define $\mathbb{N}$, does $\mathrm{ind} = \mathrm{Dim} = \dim$ on all definable sets? By 1.5, this is the same as asking whether $\mathrm{ind} = \mathrm{indcl}$ on all definable sets, but we do not yet know if $\dim = \dim\mathrm{cl}$ on all definable sets (a weaker condition, on the face of it). Indeed, we do not yet know what to say about the dimensions of complements of D_Σ sets (but we are working on it).

Are there notions of dimension that satisfy all of the properties listed in 1.2 (at least, on compact sets) but are not bounded above by Dim?

What can be said about $\mathfrak{R}$ under the assumption $\mathrm{ind} \leq \dim$ on all definable sets? As mentioned earlier, there are G_δ sets in $\mathbb{R}^3$ with $\mathrm{ind} > \dim$ ([9, 1.10.23]), so $\mathbb{N}$ is not definable (and thus Theorem A applies). But beyond this?

There is a notion dual to Dim in the literature; see Fraser [14] for information. We have not considered it here as it allows for nonempty open subsets of $\mathbb{R}$ to have dimension zero (and it is decreasing, not increasing) but perhaps there are still some associated tameness results.

We have omitted from this paper consideration of applications of Theorems A and B beyond those to the tameness program given in the introduction, in particular, we forego describing in detail obvious strengthenings of already-known results (see [26] for a few). But there are plenty of open questions to be pursued. Just one example: Suppose that E is closed and definable and $p \in \mathbb{N}$. Is there a definable C^p function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $E = f^{-1}(0)$? If $\mathfrak{R}$ defines $\mathbb{N}$, then yes, by a theorem of H. Whitney (indeed, f can then be taken to be C^∞ and real-analytic off E). If $\mathfrak{R}$ is o-minimal, then again, yes ([8, 4.22]). But in general? What if moreover $\dim = \dim\mathrm{cl}$ on all definable sets? Or if $\mathrm{ind} = \mathrm{indcl}$ on all definable sets? (And so on.)

One could investigate expansions of the ordered additive group (structures on $\mathbb{R}$ such that $\{(x, y) \in \mathbb{R}^2 : x < y\} \in \mathfrak{S}_2$ and $\mathrm{graph}(+) \in \mathfrak{S}_3$) in a spirit similar to what we have done in this paper, but the results would necessarily be rather different. To illustrate, it is known (see the end of the introduction to [20]) that if C is the classical "middle-thirds" Cantor set, then the theory of $(\mathbb{R}, <, +, C)$ is decidable (hence at least "logically tame") but $\dim_{\mathrm{H}} C = (\log 2)/(\log 3) > 0 = \mathrm{ind}\, C$. Moreover, if E is the set of midpoints of the complementary intervals of C, then E is constructible, definable and $\overline{\dim}_{\mathrm{M}} E = \overline{\dim}_{\mathrm{M}} \mathrm{cl}\, E = \overline{\dim}_{\mathrm{M}}(E \cup C) \geq \dim_{\mathrm{H}} C$. But as E is countable, $\dim_{\mathrm{H}} E = 0$. By 2.4, there are compact definable subsets of the plane for which $\mathrm{ind} < \dim$. It does seem reasonable to think that perhaps we still have $\dim_{\mathrm{H}} = \overline{\dim}_{\mathrm{P}}$ on all definable sets. (The upper packing dimension, $\overline{\dim}_{\mathrm{P}}$, of a metric space X can be defined by $\overline{\dim}_{\mathrm{P}} X = \inf_{\mathfrak{S}} \sup_{S \in \mathfrak{S}} \overline{\dim}_{\mathrm{M}} S$ where $\mathfrak{S}$ ranges over all countable covers of X. The condition $\dim_{\mathrm{H}} = \overline{\dim}_{\mathrm{P}}$ is regarded as desirable by fractal geometers.)

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