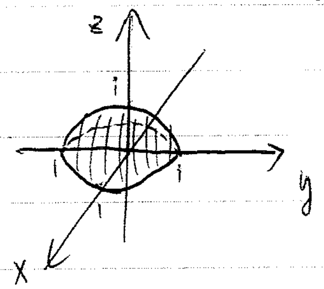
Solution for Worksheet on Nov. 30th

1. a)



b). Parameterization: $r(x,y) = (x, y, 1-x^2-y^2)$.

domain: $x^2+y^2 \le 1 \Rightarrow -1 \le x \le 1 \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$r_x \times r_y = (2x, 2y, 1)$ point upward

so $\iint F \cdot ds = \iint F \cdot (r_x \times r_y)\, dx\, dy$

$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (0, 0, 1-x^2-y^2) \cdot (2x, 2y, 1)\, dy\, dx$

$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\, dy\, dx$

change coordinate: $x = r\cos\theta$, $y = r\sin\theta$. than $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$= \int_0^{2\pi} \int_0^1 (1-r^2) r\, dr\, d\theta$

$= \frac{\pi}{2}$.

Or. $(x,y,z) = (r\cos\theta, r\sin\theta, 1-r^2)$.

then again we look at the normal vector.

$(\cos\theta, \sin\theta, -2r) \times (-r\sin\theta, r\cos\theta, 0) = (2r^2\cos\theta, 2r^2\sin\theta, r)$.

Flux $= \int_0^1 \int_0^{2\pi} (0, 0, 1-r^2) \cdot (2r^2\cos\theta, 2r^2\sin\theta, r)\, d\theta\, dr$

$= \int_0^1 \int_0^{2\pi} (1-r^2)\, r\, d\theta\, dr$

$= \frac{\pi}{2}$

c) $S_2 = \{ x^2+y^2 \le 1, \ z = 0 \}$, so $F = (0,0,0)$ on S_2,

so $\iint_{S_2} F \cdot ds = 0$.

d). $\frac{\pi}{2} + 0 = \frac{\pi}{2}$. (Because the direction of normal vector field in b) and c) are outward pointing normals).

e) $\operatorname{div} F = 1$, so from $\iiint \operatorname{div} F\, dV = \iint_S F \cdot ds$ we know that

The volume of $D = \iiint 1\, dV = \iint_S F \cdot ds = \frac{\pi}{2}$ (from (d)).

2. a) $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 2k.$

b). $\iint_{S_1} (\text{curl } F) \cdot n \, dA$

Just the same as 1(b). the difference is in 1(b) we calculate the flux of $(0,0,z)$, but here we calculate the flux of $(0,0,2)$.

$$\int_0^1 \int_0^{2\pi} (0,0,2) \cdot (2r^2\cos\theta, 2r^2\sin\theta, r) \, d\theta \, dr$$

$$= \int_0^1 \int_0^{2\pi} 2r \, d\theta \, dr$$

$$= 2\pi.$$

c). ∂S_1 is a circle with parameterization

$r(\theta) = (\cos\theta, \sin\theta, 0)$.

and the corresponding orientation of $r(\theta)$ is

θ is from 0 to 2π

then $\int F \cdot dr = \int_0^{2\pi} \underbrace{(-\sin\theta, \cos\theta, 0)}_{F \text{ on } r.} \cdot \underbrace{(-\sin\theta, \cos\theta, 0)}_{r'(\theta).} \, d\theta.$

$$= \int_0^{2\pi} 1 \, d\theta$$

$$= 2\pi$$

3 a). Volume of $D = \int_0^1 \int_0^{2\pi} \int_0^{1-r^2} 1 \cdot r \, dz \, d\theta \, dr$. (cylindrical coordinats)

$$= \frac{\pi}{2}$$

b). the surface ∂D is closed, so $\iint_{\partial D} (\text{curl } F) \cdot n \, dA = 0$.

c). $\text{div}(\text{curl } F) = 0$ (You can check it directly from the defination).