Math 347 – Exam #3 practice problems
posted November 16, 2008

Definitions etc.

1. What is meant by a *relation on a set S*?
2. When do we say that a relation on a set S is an *equivalence relation*? Be sure to carefully state the three defining properties.
3. Given an equivalence relation on a set S and an element $x \in S$, what is meant by the *equivalence class of x*?
4. What does it mean to say that the integers a and b are *congruent modulo m*? If $m = 0$, when are the integers a and b congruent modulo m?
5. If $m \in \mathbb{Z}$, what do we mean by $\mathbb{Z}_m$? If $m > 0$, how many elements are there in $\mathbb{Z}_m$?
6. How does one define addition and multiplication in $\mathbb{Z}_m$? Why does this definition make sense? What are the additive and multiplicative identities in $\mathbb{Z}_m$? Why does the distributive law hold in $\mathbb{Z}_m$? For which positive integers m is $\mathbb{Z}_m$ a field?
7. State Wilson's theorem. State Fermat's little theorem.
8. State the completeness axiom for the real numbers. Make sure you can define the term 'bounded above' when you use it.
9. What is meant by a sequence of real numbers? What does it mean to say that such a sequence *converges*? Can a sequence convergence to two different real numbers? If not, why not?
10. What does it mean for a sequence to be *increasing*? What theorem do you know about convergence of such sequences?
11. What is meant by an *infinite series*? What does it mean to say that an infinite series *converges*?
12. What does it mean to say that a sequence of real numbers is a *Cauchy sequence*?

Other problems

1. Let $S = \{1, 2, 3, 4, 5\}$. Give an example of a relation on S that is transitive and reflexive but not symmetric. Give an example of a relation on S that is transitive and symmetric but not reflexive. Give an example of a relation that is symmetric and reflexive but not transitive.
2. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$. Let $S = \{|f(x) - g(x)| : x \in \mathbb{R}\}$.
   (a) Say that f and g are related if 1 is an upper bound on the set S. Does this define an equivalence relation on the set of functions from $\mathbb{R}$ to $\mathbb{R}$? Why or why not?

(b) Say that f and g are related if the set S is bounded above. Does this define an equivalence relation on the set of functions from $\mathbb{R}$ to $\mathbb{R}$? Explain.

3. Suppose $n \in \mathbb{N}$. What are the possible last digits (in the usual decimal expansion) of n^4? (For example, if $n = 5$, then $n^4 = 625$, and so 5 is a possible last digit.) Prove that your claim is correct.

4. What is the remainder when 3^{100} is divided by 17? Use the repeated squaring method.

5. Show that there are no integers x, y, z for which $x^3 + y^3 + z^3 = 4$. *Hint:* Look at this equation modulo 9.

6. Suppose a and m are integers and $m > 0$. Prove that if $\gcd(a, m) > 1$, then $\overline{a}$ is not a unit in $\mathbb{Z}_m$. Prove that if $\gcd(a, m) = 1$, then $\overline{a}$ is a unit in $\mathbb{Z}_m$. *Hint:* Review our discussion in class about when $\mathbb{Z}_m$ is a field.

7. Suppose x and y are integers and that p is an odd prime number. Show that if $x^2 \equiv y^2 \mod p$, then either $x \equiv y \pmod{p}$ or $x \equiv -y \pmod{p}$. Deduce that the squares of the elements $\overline{1}, \overline{2}, \ldots, \overline{(p-1)/2}$ in $\mathbb{Z}_p$ are all distinct.

8. Find all prime numbers p for which $p^2 + 2$ is also prime. Prove that your answer is correct.

9. Prove that there are infinitely many natural numbers n for which $n! + 1$ is not a prime number. *Hint:* Use Wilson's theorem.

10. Prove that the set of natural numbers $\mathbb{N}$ is not bounded above. What does this have to do with the Archimedean property? (Yes, we did this in class.)

11. Prove that every convergent sequence is a Cauchy sequence.

12. Prove, directly from the definitions, that if $a_n \to L_1$ and $b_n \to L_2$, then $a_n + b_n \to L_1 + L_2$.

13. Prove, directly from the definitions, that if $a_n \to L$ and c is a positive real number, then $ca_n \to cL$.

14. Prove that if a_n is an increasing sequence of real numbers and $S = \{a_n : n \in \mathbb{N}\}$ is bounded above, then $\langle a \rangle$ converges. (This is half of the monotone convergence theorem.)

15. Suppose $\langle a \rangle$ is a convergent sequence where $a_n \to 2$ and each $a_n > 0$. Define a new sequence $\langle b \rangle$ by putting $b_n = 1/a_n$ for each n. Prove that $\lim b_n = 1/2$.

16. Show that the infinite series $\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges. What does it converge to?

17. Suppose that $\langle a \rangle$ and $\langle b \rangle$ are two sequences of nonnegative real numbers, so that $a_n \geq 0$ and $b_n \geq 0$ for every n. Suppose also that $\sum_{k=1}^{\infty} b_k$ converges. Prove that $\sum_{k=1}^{\infty} a_k$ converges.

18. Let p_k be the kth prime number, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. Does $\sum_{k=1}^{\infty} \frac{1}{p_k^2}$ converge or diverge?