

## Math 347 – Homework assignment #2

posted September 3, 2008; due Wednesday, September 10, 2008

### Additional exercises

In the following proofs you should show your steps, justifying each step from the axioms for an ordered field, as was done in class. Any results already proved from the axioms in class (such as  $a \cdot 0 = 0$ , or  $(-1)a = -a$ ) may be cited – there's no need to reprove these here.

In the first three problems below,  $F$  represents an ordered field with positive set  $P$ . (For example,  $F$  could be the real numbers.)

1. Show that if  $a, b$ , and  $c$  are three elements of  $F$  for which  $a > b$  and  $b > c$ , then  $a > c$ .
2. Show that if  $a > b$  and  $c$  is positive (i.e.,  $c$  is in the set  $P$ ), then  $ac > bc$ .
3. Prove that 1 is a positive element of  $F$ , i.e., that 1 is in  $P$ .
4. (**For fun; not to turn in**) We know from class that the real numbers  $\mathbb{R}$  are a complete, ordered field. This exercise outlines why these properties are enough to imply that 2 has a square root in  $\mathbb{R}$ . One can adapt this argument to show that every nonnegative real number has a real square root.

Define  $S = \{x \in \mathbb{R} : x^2 < 2\}$ . (Here the symbol '2' stands for  $1 + 1$ .)

- (a) Show that  $S$  is a nonempty subset of  $\mathbb{R}$  that is bounded above. (In fact, you should be able to prove that 2 is an upper bound for  $S$ .)
- (b) The completeness axiom for  $\mathbb{R}$  guarantees that  $S$  has a least upper bound  $M$  (so  $M$  is an upper bound for  $S$  but no number less than  $M$  is an upper bound for  $S$ ).

Prove that this number  $M$  satisfies  $M^2 \geq 2$ . Show also that this  $M$  satisfies  $M^2 \leq 2$ . Deduce that  $M^2 = 2$ . (*Hint:* To show  $M^2 \geq 2$ , assume for the sake of contradiction that  $M^2 < 2$  and deduce that  $M$  is too small to be an upper bound on  $S$ . To show  $M^2 \leq 2$ , assume that  $M^2 > 2$ , and argue that  $M$  is too big to be the *least* upper bound.)

Part (a) should be doable and, while not assigned, is the sort of problem you should know how to do. Part (b) is considerably harder and involves thought processes from analysis; **don't worry if you don't see how to do this!**