Math 347 – Homework #4 solutions
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Solutions

Exercise 4.5. Such a bijection exists precisely when A has at least two elements. If A is the empty set or if $A = \{1\}$, then the identity map is the *only* function from A to A, and so certainly the only bijection. Suppose now that A has at least two elements. Pick distinct elements $a_1, a_2 \in A$, and define $f\colon A \to A$ by setting, for $x \in A$,

$$f(x) := \begin{cases} a_2 & \text{if } x = a_1, \\ a_1 & \text{if } x = a_2, \\ x & \text{if } x \neq a_1, a_2. \end{cases}$$

Let us check that f is a bijection.

Injective: Suppose that x_1, x_2 are elements of A for which $f(x_1) = f(x_2)$. We will prove that $x_1 = x_2$. First, notice that by our definition of f, the only element that maps to a_2 is a_1, and vice versa. So we may suppose that neither x_1 nor x_2 is a_1 or a_2. But in that case $f(x_1) = x_1$ and $f(x_2) = x_2$, so that from $f(x_1) = f(x_2)$ we deduce immediately that $x_1 = x_2$.

Surjective: We have to show that every element $x \in A$ is in the image of f. Certainly a_1 is in the image of f, since $a_1 = f(a_2)$, and similarly a_2 is in the image of f since $a_2 = f(a_1)$. If x is an element of A other than a_1 or a_2, then by our definition of f, we have $f(x) = x$.

Exercise 4.8 We have $(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = (x^2 - 1) - 1 = x^2 - 2$, and $(g \circ f)(x) = g(f(x)) = f(x)^2 - 1 = (x - 1)^2 - 1 = x^2 - 2x$.

Exercise 4.21 We define a function f from A to B by setting, for S an element of A,

$$f(S) := \begin{cases} S \setminus \{n\} & \text{if } n \in S, \\ S \cup \{n\} & \text{if } n \notin S. \end{cases}$$

Notice that since in each case we either remove an element from S or add an element to S, we obtain from S (which started with an even number of elements) a set with an odd number of elements. Thus f is well-defined. Let us check that f is a bijection:

Injective: Suppose that $S_1, S_2 \in A$ and $f(S_1) = f(S_2)$. Since $f(S_1) = f(S_2)$, either n is in both S_1 and S_2 or n is in neither S_1 nor S_2: indeed, if n is in one but not the other, then by our definition of f, exactly one of the sets $f(S_1), f(S_2)$ contains n, contradicting that $f(S_1) = f(S_2)$. Consider first the case when n is in both S_1 and S_2. Then $f(S_1) = S_1 \setminus \{n\}$ and $f(S_2) = S_2 \setminus \{n\}$. By hypothesis, $f(S_1) = f(S_2)$, so that $S_1 \setminus \{n\} = S_2 \setminus \{n\}$. But then adding n back to both sets we see that $S_1 = S_2$. If we are in the case when neither S_1 nor S_2 contains n, then the equation $f(S_1) = f(S_2)$ says that $S_1 \cup \{n\} = S_2 \cup \{n\}$. In this case $S_1 = S_2$ follows from removing n from both sides.

Surjective: We have to show that every element of B has the form $f(S)$ for some $S \in A$. Suppose T is an element of B, i.e., that T is a subset of $[n]$ with an even number of element. If T contains n, consider the set S obtained by removing n from T. This set S has an odd number of elements, so that $S \in A$, and by our definition of f, we have $f(S) = T$. On the other hand, if T does not contain n, consider the subset S obtained by adding the element n to T. Then S has an odd number of elements, so that $S \in A$, and by our definition of f we have $f(S) = T$. So in either case we have shown there is some $S \in A$ for which $f(S) = T$. Since T was an arbitrary element of B, this proves that f is surjective.

Exercise 4.31. We have to prove that for every pair of elements $b_1, b_2 \in B$ for which $b_1 < b_2$, we have $f^{-1}(b_1) < f^{-1}(b_2)$. Let $a_1 = f^{-1}(b_1)$ and $a_2 = f^{-1}(b_2)$, so that we want to show $a_1 < a_2$. Suppose for the sake of contradiction that $a_1 \geq a_2$. If $a_1 = a_2$ then $f(a_1) = f(a_2)$. However, $f(a_1) = f(f^{-1}(b_1)) = b_1$ and $f(a_2) = f(f^{-1}(b_2)) = b_2$, so that $f(a_1) = f(a_2)$ contradicts $b_1 < b_2$. If $a_1 > a_2$, then $b_1 = f(a_1) > f(a_2) = b_2$, because we are given that f is increasing on A. But $b_1 < b_2$, so again we have a contradiction. So our assumption that $a_1 \geq a_2$ must be false. Hence $a_1 < a_2$.

4.34 (a) This is true. Indeed, suppose $f(x_1) = f(x_2)$. Applying g to both sides of this inequality, we deduce that

$$h(x_1) = (g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2) = h(x_2).$$

Since h is injective, this inequality implies that $x_1 = x_2$. We have proved that whenever $f(x_1) = f(x_2)$, we have $x_1 = x_2$, thus verifying the definition of injectivity.

(b) This is false. Suppose $A = \{1\}$ and $B = C = \mathbb{R}$. Let $f\colon A \to B$ be the function defined by $f(1) = 1$, and let $g\colon B \to C$ be the function defined for $x \in B$ by $g(x) = x^2$.

We know that $h = g \circ f$ is a function from A to C. Since the domain A of h contains only one element, h is automatically injective. However, g is not injective, since $g(-2) = g(2) = 4$.

(c) This is false. Take $A = B = \mathbb{R}$, and take $C = \{1\}$. Let $f \colon A \to B$ be defined by $f(x) = x^2$ for $x \in A$, and let $g \colon B \to C$ be defined by $g(y) = 1$ for $y \in B$. We know that $h = g \circ f$ is a function from A to C. Since the domain of h is nonempty and the target of h has only one-element, h is automatically surjective. However, f is not surjective. Indeed, -1 is an element of B, but there is no $x \in A$ for which x^2 is -1.

(d) This is true. Let c be any element of C. Since $h \colon A \to C$ is surjective, there is an $a \in A$ for which $h(a) = c$. Since $h = g \circ f$, we have that $g(f(a)) = c$. Hence c is in the image of g. Since c was an arbitrary element of C, it follows that g is surjective.

Exercise 4.37. This is the special case of Exercise 4.34(a) where $A = B = C$ and $g = f$.

Extra prob #1 (a) Consider the statement

$$P(n)\colon \text{If there is an injective function from } [n] \text{ to } [m], \text{ then } n \leq m.$$

We wish to prove that $P(n)$ is true for all nonnegative integers n. First notice that $P(0)$ is true: $P(0)$ says that if there is an injective function from $[0]$ to $[m]$, then $m \geq 0$. But we are given that m is a nonnegative integer, so automatically $m \geq 0$; there is nothing to prove here. So it's enough to consider the statements $P(n)$ for $n \geq 1$. That is, it is enough to prove $P(n)$ for every natural number n. This suggests we use induction.

Base case: $P(1)$ says that if there an injective function from $[1]$ to $[m]$, then $m \geq 1$. Since m is a nonnegative integer, the statement $m \geq 1$ is true except when $m = 0$. So it is enough to show that there is no injective function from $[1]$ to $[0]$. But $[0] = \emptyset$, and there is no function at all (let alone an injective one) from a nonempty set (such as $[1]$) to the empty set.

Inductive step: We have to show that for all $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$. Suppose that $P(n)$ is true. To study $P(n+1)$, suppose we are given an injective map $f \colon [n+1] \to [m]$. We would like to prove that

$$n + 1 \leq m.$$

Since f is injective, the element $f(n+1)$ is not the image under f of any element in $[n]$. Hence there is a well-defined function $\tilde{f} \colon [n] \to [m] \setminus$

$\{f(n+1)\}$ given by putting $\tilde{f}(x) = f(x)$ for every $x \in [n]$. Moreover, $\tilde{f}$ is injective, since $\tilde{f}$ is obtained by restricting f and we are given that f is injective. Also, from class we know there is a bijection

$$g\colon [m] \setminus \{f(n+1)\} \to [m] \setminus \{m\} = [m-1].$$

Since both g and $\tilde{f}$ are injective, the composition $g \circ \tilde{f}\colon [n] \to [m-1]$ is also injective. From the induction hypothesis $P(n)$, we deduce that $n \leq m-1$. Rearranging this inequality gives us $n+1 \leq m$, which is what we wanted to show.

(b) Consider the statement

$P(n)$: If there is a surjective function from $[n]$ to $[m]$, then $n \geq m$.

Again this is trivially true if $n = 0$. To see this, notice that any function whose domain is the empty set must have empty image (simply because there are no elements in the domain to map). So if $[m]$ is not empty, there cannot be a surjective function from $[0] = \emptyset$ to $[m]$. In other words, if there is a surjective function from $[0]$ to $[m]$, then $[m]$ is empty, so that $m = 0$. So again it is enough to consider the case of natural numbers n, and again we proceed by induction.

Base case: $P(1)$ claims that if there is a surjective function from $[1]$ to $[m]$, then $m \leq 1$. Suppose for the sake of contradiction that we have a surjection $f\colon [1] \to [m]$ with $m > 1$. Since the domain of f is the one-element set $\{1\}$, the image of f is the one-element set $\{f(1)\}$. If $f(1) = m$, then the image of f fails to contain the element $m-1 \in [m]$, so that f is not surjective. If $f(1) \neq m$, then the image of f fails to contain the element $m \in [m]$, so that again f is not surjective. So in either case f is not surjective. This contradiction establishes that $m \leq 1$.

Induction step: Suppose $P(n)$ is true. To study $P(n+1)$, assume we have a surjective function $f\colon [n+1] \to [m]$. We want to prove that

$$n+1 \geq m.$$

As in the proof of (a), we consider what happens when we restrict the domain of f to the set $[n]$, forgetting about the element $n+1 \in [n+1]$.

There are two cases: It might be that every element of $[m]$ is the image of an element of $[n]$ under f. In this case we get a surjective map $\tilde{f}\colon [n] \to [m]$ just by putting $\tilde{f}(x) = f(x)$ for $x \in [n]$. Then by the induction hypothesis, we have $n \geq m$, which is even stronger than saying $n+1 \geq m$.

In the remaining case, not every element of $[m]$ is the image under f of an element of $[n]$. Since $f\colon [n+1] \to [m]$ is surjective, it must be that the image of $[n]$ under f is only missing the single element $f(n+1)$. So in this case we get a surjection if we define $\tilde{f}\colon [n] \to [m] \setminus \{f(n+1)\}$ by putting $\tilde{f}(x) = f(x)$ for $x \in [n]$. Once again, we know there is a bijection

$$g\colon [m] \setminus \{f(n+1)\} \to [m] \setminus m = [m-1].$$

Since g is surjective and $\tilde{f}$ is surjective, this gives us a surjective map $g \circ \tilde{f}\colon [n] \to [m-1]$. By the induction hypothesis, $n \geq m-1$. Rearranging this inequality gives us $n+1 \geq m$, which is what we wanted to prove.

(c) This follows immediately from parts (a) and (b) and the definition of a bijection.

Extra prob #2 Consider the statement

$P(n)$: If A is a finite set of size n, then every proper subset B of A has size m for some $m < n$.

The statement $P(0)$ is vacuously true: if A has size 0, then A is empty, and so has no proper subsets. So it is enough to show that $P(n)$ is true for every natural number n. Again we apply induction.

Base case: $P(1)$ says that if A is a finite set of size 1, then every proper subset of A has size m for some $m < 1$. Suppose A has size 1. By our definition of size, it follows that there is a bijection $f\colon A \to [1]$. The inverse map $f^{-1}\colon [1] \to A$ is then also a bijection. It follows that $A = \{f^{-1}(1)\}$. In particular, A has only one element, so that the only proper subset of A is the empty set. The empty set has size 0 and $0 < 1$. This completes the proof of $P(1)$.

Induction step: Suppose $P(n)$ is true. To analyze $P(n+1)$, suppose A is a set of size $n+1$ and B is a proper subset of A. We want to show that B has size strictly less than $n+1$.

This is easy if B is empty, since then B has size 0 and $0 < n+1$. So let's suppose B is not empty, and let's choose an element $b \in B$. Since B is a proper subset of A, it follows that $B \setminus \{b\}$ is a proper subset of $A \setminus \{b\}$. But $A \setminus \{b\}$ is a set of size n. (Recall that we proved in class that taking away a single element from a finite set decreases the size by 1.) It now follows from the induction hypothesis $P(n)$ that $B \setminus \{b\}$ has size m' for some integer $0 \leq m' < n$. But then

$$|B| = |(B \setminus \{b\}) \cup \{b\}| = |B \setminus \{b\}| + |\{b\}| = m' + 1.$$

Thus B has size $m' + 1 < n + 1$, as desired.

Extra prob #3 Suppose for the sake of contradiction that A is finite, with $|A| = n$. Since B is a proper subset of A, from extra problem #2 we know that $|B| = m$ for some $m < n$. Unwrapping our definition of **size**, we see there are bijections $f: A \to [n]$ and $g: B \to [m]$. Moreover, we are supposing there is an injective map, say h, which goes from A to B.

To get a contradiction we consider the composite map $g \circ h \circ f^{-1}: [n] \to [m]$. Since g, h and f^{-1} are all injective maps, it follows that the composition $g \circ h \circ f^{-1}$ is also injective. So we have an injective map from $[n]$ to $[m]$, where $m < n$. This contradicts part (a) of extra problem #1.

It follows that our assumption that A is finite must be wrong; in other words, A is infinite.