Math 347 – Homework #8 solutions
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Solutions

Exercise 13.6 The flaw in this argument is that not every number in $(0, 1)$ shows up on the list. Indeed, consider the number $1/3 = 0.333333\ldots$. Since this number has no last nonzero digit, it will not be listed in the nth step for any n, and so will not show up at all!

Exercise 13.11 The assertion in (a) is true. Indeed, suppose $a_n \to L_1$ and $b_n \to L_2$, where $L_1 < L_2$. Let ϵ be a positive number for which $\epsilon < (L_2 - L_1)/2$. Choose natural numbers N_0, N_1 with the properties that $|a_n - L_1| < \epsilon$ if $n \geq N_0$ and $|b_n - L_2| < \epsilon$ if $n \geq N_1$, and put $N = \max\{N_0, N_1\}$. Then if $n \geq N$, we have $|a_n - L_1| < \epsilon$, so that $a_n < L_1 + \epsilon < (L_1 + L_2)/2$ (by our choice of ϵ). But we also have $|b_n - L_2| < \epsilon$, so that $b_n > L_2 - \epsilon > (L_1 + L_2)/2$. Hence $a_n < (L_1 + L_2)/2 < b_n$ for all $n \geq N$.

The assertion of part (b) is false: For example, suppose $a_n = 1/n$ for all n and $b_n = 0$ for all n. Then $\lim a_n = \lim b_n = 0$, but $a_n > b_n$ for every value of $n \in \mathbb{N}$.

Exercise 13.23 Let $\alpha = \sup A$ and $\beta = \sup B$. If $x \in A$, then $x \leq \alpha$, and if $y \in B$, then $y \leq \beta$. Hence $x + y \leq \alpha + \beta$. Since every element of C has the form $x + y$ where $x \in A$ and $y \in B$, it follows that $\alpha + \beta$ is an upper bound for C. To see that it is the least upper bound, we use Proposition 13.15. By that proposition, there is a sequence $\langle x \rangle$ with each $x_n \in A$ and with $x_n \to \alpha$. Also, there is a sequence $\langle y \rangle$ with each $y_n \in A$ and with $y_n \to \beta$. Then $x_n + y_n \in C$ for every n and $x_n + y_n \to \alpha + \beta$. Thus $\alpha + \beta$ is an upper bound on C, and there is a sequence from C converging to $\alpha + \beta$. Referring again to Proposition 13.15 shows that $\alpha + \beta = \sup C$.

Exercise 13.28 For us to assert that "$\lim x_n y_n = \lim(x_n) \lim(y_n)$" we need to know that both $\lim x_n$ and $\lim y_n$ exist. We know that $\lim x_n$ exists, but nothing in the problem statement requires that $\lim y_n$ exists. (For example, if $y_n = (-1)^n$, then $|y_n| \leq 1$ for each n, but $\langle y \rangle$ does not converge.)

A correct proof is as follows: Let $\epsilon > 0$. Choose $N_0 \in \mathbb{N}$ so that $|x_n| < \epsilon$ if $n \geq N_0$. (This is possible since $x_n \to 0$.) For any $n \geq N_0$, we have $|x_n y_n| = |x_n||y_n| \leq |x_n| < \epsilon$ (since $|y_n| \leq 1$). Hence $x_n y_n \to 0$.

Exercise 13.30 We first show that the sequence x_n is nondecreasing, i.e., that $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$. Writing out both x_n and x_{n+1}, we find that

$$\begin{aligned} x_{n+1} - x_n &= \left(\frac{1}{2n+1} + \frac{1}{2n+2} \right) - \frac{1}{n+1} \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} > 0, \end{aligned}$$

so that $x_{n+1} > x_n$. Let $S = \{x_n : n \in \mathbb{N}\}$. Since $x_n < \frac{n}{n+1} < 1$, the set S is bounded above (with 1 as an upper bound). So $\langle x \rangle$ converges by the monotone convergence theorem.

Problem 6 Suppose that L is the common limit of a_n and c_n. Pick $N_0 \in \mathbb{N}$ so that $|a_n - L| < \epsilon$ whenever $n \geq N_0$, and pick $N_1 \in \mathbb{N}$ so that $|c_n - L| < \epsilon$ whenever $n \geq N_1$. Put $N_2 = \max\{N_0, N_1\}$. If $n \geq N_2$, then $b_n - L \geq a_n - L > -\epsilon$. But also $b_n - L \leq c_n - L < \epsilon$. Since $-\epsilon < b_n - L < \epsilon$, we have $|b_n - L| < \epsilon$. Thus $b_n \to L$.

Problem 7 Let $\epsilon > 0$. Since $\sqrt{x}$ is increasing for $x > 0$, we have $1 = \sqrt{1} \leq \sqrt{1 + n^{-1}}$ for each $n \in \mathbb{N}$. We also know that $\sqrt{1 + n^{-1}} < 1 + n^{-1}$ for every $n \in \mathbb{N}$. (Indeed, $1 + n^{-1} < 1 + 2n^{-1} + n^{-1} = (1 + n^{-1})^2$, and the last inequality follows upon taking the square root of both sides.) Thus $0 \leq \sqrt{1 + n^{-1}} - 1 < n^{-1}$. By the Archimedean property we can choose $N_0 \in \mathbb{N}$ so that $N_0^{-1} < \epsilon$. Then if $n \geq N_0$, we have $|\sqrt{1 + n^{-1}} - 1| < n^{-1} \leq N_0^{-1} < \epsilon$. This proves that $\lim \sqrt{1 + n^{-1}} = 1$, as desired.

Problem 8 Assuming S is a nonempty subset of $\mathbb{R}$ that is bounded below, put $S' = \{-x : x \in S\}$. Then S' is bounded above; indeed, if $x \geq M$ for every $x \in S$, then $y \leq -M$ for every $y \in S'$. Moreover, S' is nonempty since S is nonempty. By the completeness axiom, S has a least upper bound α. Set $\beta = -\alpha$. We claim that β is the greatest lower bound of S.

First notice that β is a lower for S: If $x \in S$, then $-x \in S'$, and so $-x \leq \alpha$, so that $x \geq -\alpha = \beta$. Moreover, β must be the greatest lower bound of S: if $\gamma > \beta$ is a lower bound for S, then $x \geq \gamma$ for all $x \in S$, so that $y \leq -\gamma$ for every $y \in S'$. But then $-\gamma$ is an upper bound for S'. This is impossible since $-\gamma < -\beta = \alpha$ and α is the least upper bound of S'.

Problem 9 We will prove that if $a < b$ are *any two* real numbers, then there is a rational number x between a and b. First observe that if $A < B$ are two real numbers for which $B - A > 1$, then there is an integer n with $A < n < B$. Pick a natural number $m \in \mathbb{N}$ for which $m(b - a) > 1$. (This is possible by the Archimedean property.) Then if we set $a = mA$ and $B = mb$, we have $B - A > 1$, and so there is an integer n with $ma = A < n < B = mb$. But then $a < n/m < b$. So n/m is a rational number with the desired property.