

**Math 595 – Homework assignment #2**  
posted February 12, 2009; due Monday, February 23, 2009

**Exercises**

Do 6 of the following 8 exercises, or more for extra credit.

Exercises 4 and 5 now appear as Exercises 25 and 26 of the text; the numbering of the other exercises is unchanged.

1. Exercise 15.
2. Exercise 18.
3. Exercise 19.
4. In this exercise and the next we explore what can be proved with our present tools about the magnitude of the divisor function  $\tau(n)$ .

(a) Show that  $\sum_{n \leq x} \tau(n) = x \log x + O(x)$ . So on average, an integer  $n \leq x$  has about  $\log x$  divisors.

(b) Show that  $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)}$  for every natural number  $n$ . Deduce from Exercise 18 that for each  $B > 0$ , all but  $O(x/B^2)$  of the natural numbers  $n \leq x$  have

$$2^{\log \log x - B\sqrt{\log \log x}} \leq \tau(n) \leq 2^{\log \log x + B\sqrt{\log \log x}}.$$

Since  $2^{\log \log x} = (\log x)^{\log 2}$ , this shows that most  $n \leq x$  have significantly fewer divisors than the average.

5. Let  $n$  be a natural number not exceeding  $x$ . Put  $A = \prod_{p^e \parallel n, p \leq \frac{\log x}{(\log \log x)^2}} p^e$  and put  $B = \prod_{p^e \parallel n, p > \frac{\log x}{(\log \log x)^2}} p^e$ .

(a) Show that  $\tau(A) \leq 2^{O(\log x / (\log \log x)^2)}$ .

(b) Show that  $\Omega(B) \leq (1 + o(1)) \log x / \log \log x$ . Deduce that  $\tau(B) \leq 2^{(1+o(1)) \frac{\log x}{\log \log x}}$ .

(c) Conclude from (a) and (b) that  $\tau(n) \leq 2^{(1+o(1)) \frac{\log x}{\log \log x}}$  as  $x \rightarrow \infty$ .

(d) By considering the product of an initial segment of the primes, show that there is a sequence of  $n$  tending to infinity along which

$$\tau(n) \geq 2^{(1+o(1)) \frac{\log n}{\log \log n}}.$$

Thus (c) is best possible. You *may* assume the prime number theorem for this part of the exercise, but it is possible to avoid this.

6. Exercise 22.
7. Exercise 23.
8. Exercise 24.