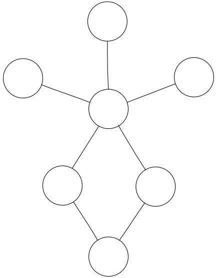
Environmental Evolutionary Graph Theory

Gregory Puleo
gjp6297@rit.edu

Rochester Institute of Technology
REU Host: University of Illinois at Urbana-Champaign

MIGHTY XLVII
November 8, 2008

What Is Evolutionary Graph Theory?



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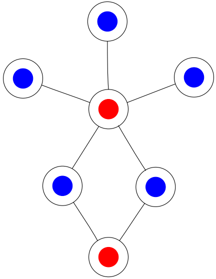

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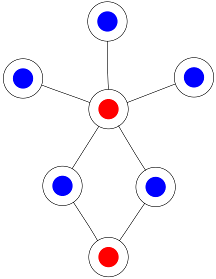

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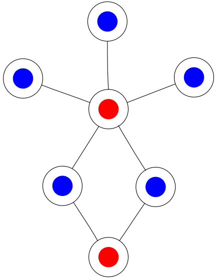

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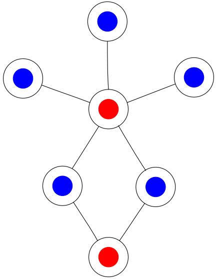

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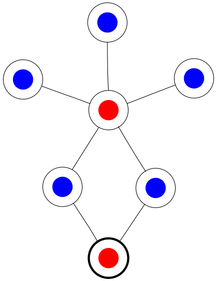

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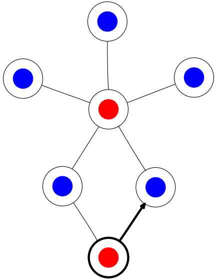

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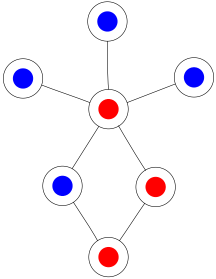

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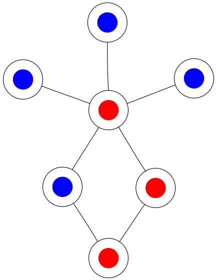
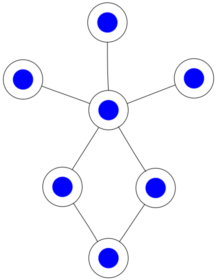
Fixation



- This model is a stationary Markov chain.

Fixation



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- The all-blue and all-red states are absorbing.

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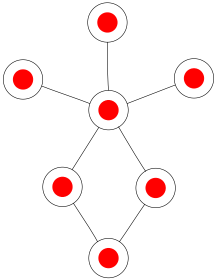

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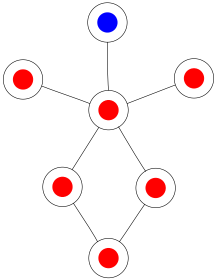

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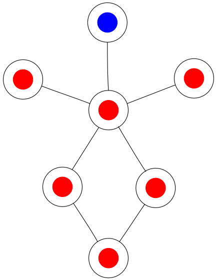
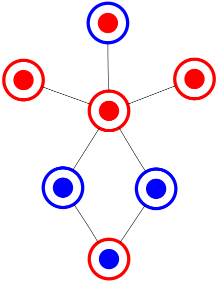


- This model is a stationary Markov chain.
- The all-blue and all-red states are absorbing.
- When only one species remains, that species has achieved fixation.
- We will frequently be interested in the pr. that a randomly placed individual of one color achieves fixation:
  - ρ_{R} – the pr. that a single red individual, placed uniformly randomly with everyone else blue, achieves fixation
  - ρ_{B} – the pr. that a lone blue individual achieves fixation

Adding the "Environmental"



- We give the vertices background colors.

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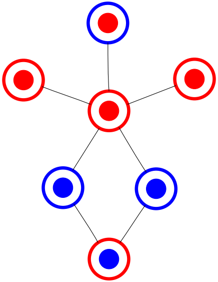

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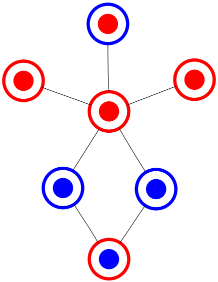

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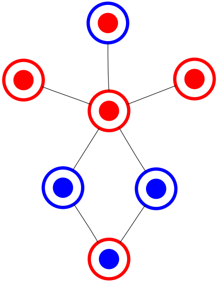

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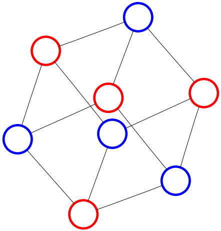
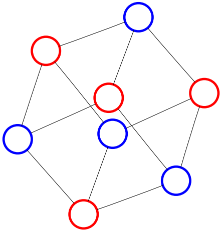
- As before, an individual is picked for reproduction with pr. proportional to its fitness.

Cubes and Damaged Cubes



- By symmetry, this cube graph must be fair:

Cubes and Damaged Cubes



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$$\rho_{\mathrm{R}} = \rho_{\mathrm{B}} = 1/8$$

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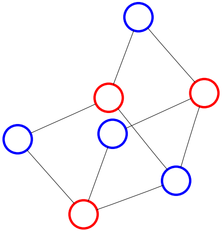

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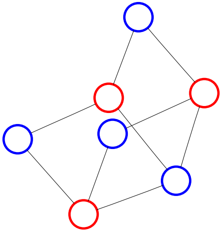

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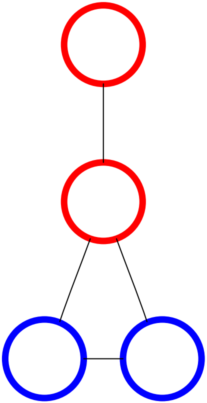
- What about a damaged cube?
- Damaged cubes are also fair:

$$\rho_{\mathrm{R}} = \rho_{\mathrm{B}} = 1/7$$

Hangers are Unfair



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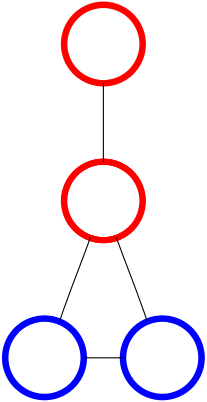

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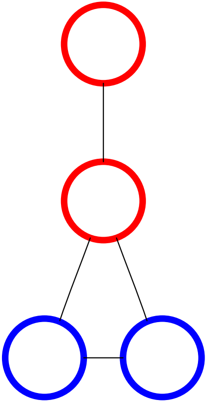
- This “hanger” graph favors red:

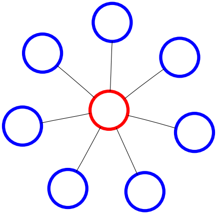
Hangers are Unfair



- On the other hand, many graphs give one color an advantage.
- This "hanger" graph favors red:

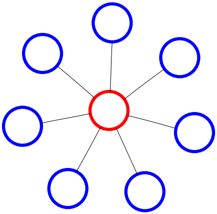
ε	ρ_{R}	ρ_{B}
.9	.256	.243
.5	.284	.201
.1	.319	.080

A Star Graph



- This star graph is fair.

A Star Graph



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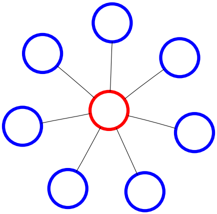

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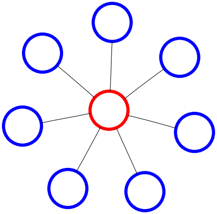
Question

Can we characterize fair graphs?

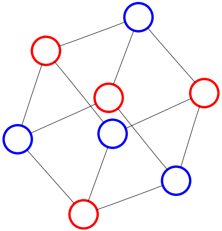
Properly Two-Colored Graphs



Definition

A graph is properly two-colored if no two vertices with the same background color are adjacent.

Properly Two-Colored Graphs



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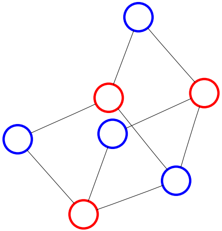

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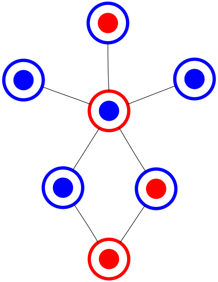


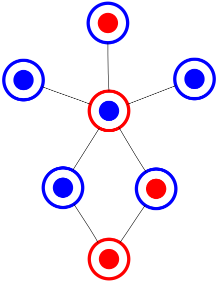
Definition

A graph is properly two-colored if no two vertices with the same background color are adjacent.

Theorem

Let G be a properly two-colored graph, let S be any state of G, let v be any vertex of G, and let w be any opposite-color neighbor of v. Then $\phi(v) = \phi(w)$.

Notation for the Fixation Probability Vector



- Let $\vec{x}$ be a 2^n-vector where the entry x_s is the pr. red fixates starting from the state S.

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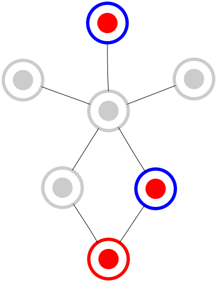

- Let $\vec{x}$ be a 2^n-vector where the entry x_S is the pr. red fixates starting from the state S.
- Identify each state S with the set of vertices containing red individuals.

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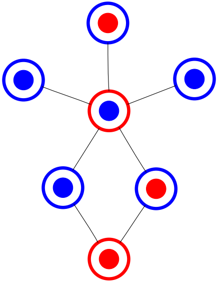

- Let $\vec{x}$ be a 2^n-vector where the entry x_s is the pr. red fixates starting from the state S.
- Identify each state S with the set of vertices containing red individuals.
- Define the quantity γ by

$$\gamma = \frac{1}{\sum_{v \in V(G)} \frac{1}{\deg(v)}}$$

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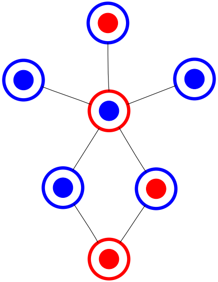

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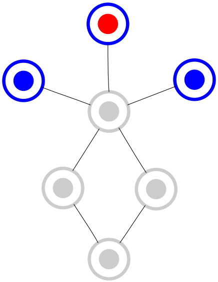

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$$\gamma = \frac{1}{\sum_{v \in V(G)} \frac{1}{\deg(v)}}$$
$$= \frac{1}{3\left(\frac{1}{1}\right)}$$

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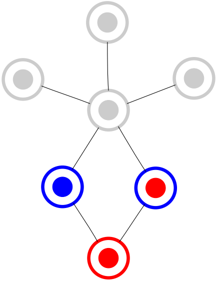

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$$
\begin{aligned}
\gamma &= \frac{1}{\sum_{v \in V(G)} \frac{1}{\deg(v)}} \\
&= \frac{1}{3\left(\frac{1}{1}\right) + 3\left(\frac{1}{2}\right)}
\end{aligned}
$$

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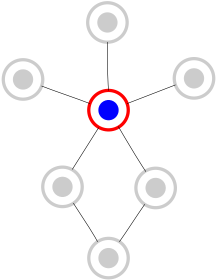

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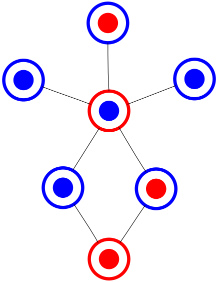

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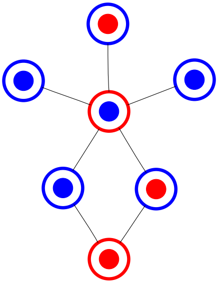
$$\gamma = \frac{1}{\sum_{v \in V(G)} \frac{1}{\deg(v)}} = \frac{1}{3\left(\frac{1}{1}\right) + 3\left(\frac{1}{2}\right) + 1\left(\frac{1}{5}\right)} = \frac{10}{47}$$

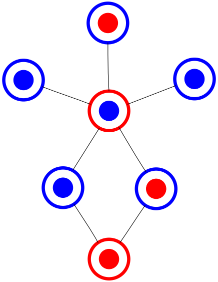


Theorem

For a properly two-colored graph G,

$$x_S = \gamma \sum_{v \in S} \frac{1}{\deg(v)}$$

A Value for the Fixation Probability Vector



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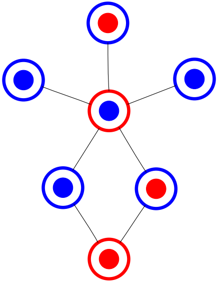
- For the current state of the graph at left, this yields

$$x_S = \frac{10}{47}\left(1\left(\frac{1}{1}\right) + 2\left(\frac{1}{2}\right)\right) \approx .426$$

A Value for the Fixation Probability Vector



Theorem

For a properly two-colored graph G,

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Corollary

All properly two-colored graphs are fair.

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Proving Two-Colored Graphs are Fair

Proof.

- For each vertex v, let $y_v = x_{\{v\}}$.

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- Now, consider this formula for ρ_{R}:

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$$y_v = \frac{\gamma}{\deg(v)} = \frac{\frac{1}{\deg(v)}}{\sum_{w \in V(G)} \frac{1}{\deg(w)}}$$

- Now, consider this formula for ρ_{R}:

$$\rho_{\mathrm{R}} = \frac{1}{n} \sum_{v \in V(G)} y_v$$

$$\rho_{\mathrm{B}} = \rho_{\mathrm{R}} = \frac{1}{n}$$

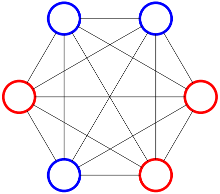
$\square$

Does the Converse Hold?

Proposition

All fair graphs are properly two-colored.

Does the Converse Hold?



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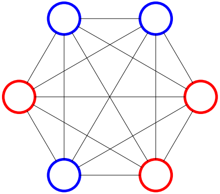

~~Proposition~~

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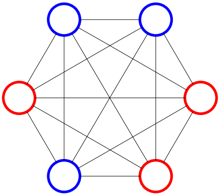

~~Proposition~~

~~All fair graphs are properly two-colored.~~

Open Question

What is the general rule characterizing fair graphs?

Future Work

Open Question

Is there a nice closed-form expression for the entries of $\vec{x}$ on arbitrary graphs?

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Is there an efficient algorithm for numerically calculating fixation probabilities?

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Is there an efficient algorithm for numerically calculating fixation probabilities?

Generalizations and Extensions

- Different types of graph
  - Directed graphs
  - Weighted edges
- Different "ecology"
  - Species-specific fitness
  - More than two foreground/background colors

Further Reading

E. Lieberman, C. Hauert, and M.A. Nowak.
Evolutionary dynamics on graphs.
Nature **433** (2005), no. 7023, 312–316

M.A. Nowak.
Evolutionary Dynamics.
Cambridge: Belknap Press, 2006.