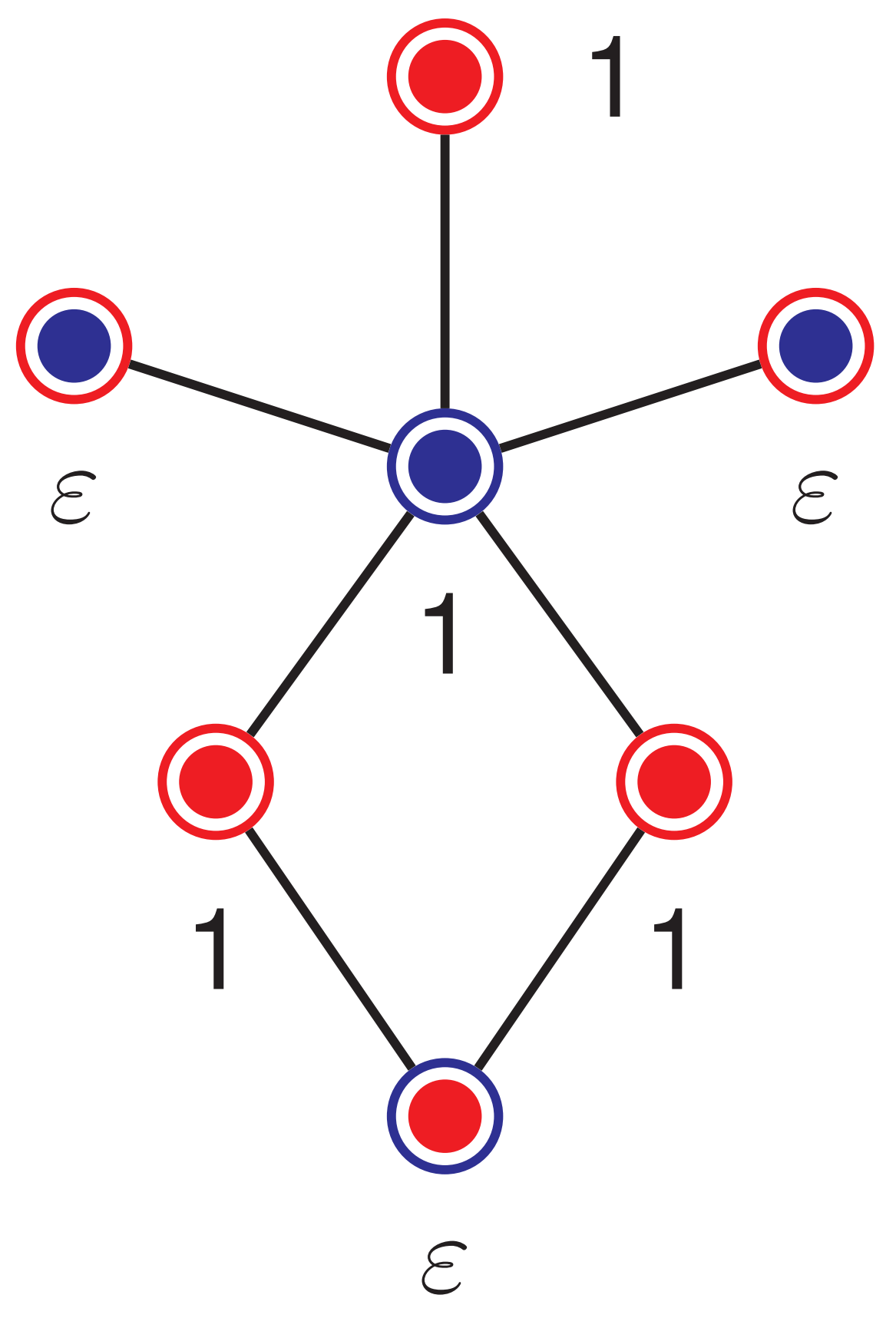
# Environmental Evolutionary Graph Theory

## I. Motivation

- Two types of island: red and blue, connected by landbridges.
- Two animal species: red and blue.
- Red islands favor red animals and vice versa.
- Central question: how do these low-level selection pressures determine which species is favored by the archipelago as a whole?

## II. Graph Model



- Each vertex has a *background* and *foreground* color.
  - Background: fixed, represents terrain
  - Foreground: changes with time, represents dominant species
- The *fitness* of a vertex at a particular time is 1 if its foreground color matches its background color, and some fixed $\varepsilon > 0$ if it does not.

## III. Dynamics

- Time is *discrete*: $t = 1, 2, 3, \ldots$
- During each time unit, we update the model state as follows:
  - Pick a vertex $v$ for *reproduction*, with probability proportional to its fitness.
  - Pick one of its neighbors $w$ for *death*, with uniformly distributed probability.
  - Overwrite the foreground color of $w$ with the foreground color of $v$.
- This update rule defines a *Markov chain* whose state space is the set of all foreground colorings of $G$.
- The all-red and all-blue states are *absorbing*: once we reach one of them, we cannot escape.
- Given any initial setup, with probability 1 we eventually end up in either the all-blue state or the all-red state.

## IV. Questions

1. Given a graph (with *only* its background coloring), can we determine whether it “favors” one species or the other? Can this question be made meaningful?
2. Given an arbitrary initial state, can we compute the probability that we will end up in the all-red state?

These questions appear hard to answer in general, but for a specific class of graphs they become surprisingly easy.

## V. Definitions

A graph is *properly two-colored* if every edge joins two vertices with different background colors. For example, the graph at left is properly two-colored.

Given a graph $G$, define its associated mysterious constant $\gamma_G$ by

$$\gamma_G = 1 \Bigg/ \sum_{v \in V(G)} \frac{1}{\deg v}.$$

## VI. Main Theorem

Let $G$ be a properly two-colored graph, and let $R \subset V(G)$ be given. Let $x_R$ denote the probability of ending up in the all-red state, given that we start from the state where only the vertices in $R$ are red. Then

$$x_R = \gamma_G \sum_{w \in R} \frac{1}{\deg w}.$$

## VII. Example

For the graph in §II, we have

$$\gamma_G = 1 \Bigg/ \left(\frac{3}{1} + \frac{3}{2} + \frac{1}{5}\right) = \frac{10}{47},$$

so that when $R$ is as pictured,

$$x_R = \frac{10}{47}\left(\frac{1}{1} + \frac{3}{2}\right) = \frac{25}{47} \approx 0.53.$$

## VIII. Corollary

Two-colored graphs are *fair*, in the following sense: let $\rho_R$ denote the probability that a single red individual – placed randomly on the graph with everyone else blue – succeeds in taking over. Define $\rho_B$ symmetrically. If $G$ is two-colored, then

$$\rho_R = \rho_B = 1/\#V(G).$$

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