

Exam 1

Math 241

June 21, 2007

Name: KEY

Instructions:

1. You have 90 minutes to finish the test. It should take around one hour. The exam is worth 100 points.
2. There are no calculators allowed on the test.
3. Answers involving inverse trigonometric functions are acceptable, especially if the answer does not simplify to a reference angle.
4. No ipods or music systems are allowed during the test.
5. No cell phones are allowed during the test.

1. (4 pts) Define what it means for two lines to be skew.

Two lines are skew if they are not parallel and nonintersecting.

2. (6 pts) State the formula for curvature and give the curvature of the circle $(x - 2)^2 + y^2 = 9$.

$$K = \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{v} \left| \frac{d\vec{T}}{dt} \right| = \frac{|\vec{v} \times \vec{a}|}{v^3}$$

Circle: $r = 3, \quad K = \frac{1}{3}$

3. (4 pts) State a formula for the magnitude of the cross product of two vectors in terms of the angle between the vectors.

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

4. (6 pts) State the equations that give x , y and z in terms of ρ , ϕ and θ .

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

5. (7 pts) Find the angle between the vectors $\langle 5, -5, 5 \rangle$ and $\langle 2, 0, 0 \rangle$. You may not be able to solve for an actual number.

$$\langle 5, -5, 5 \rangle \cdot \langle 2, 0, 0 \rangle = 10$$

$$|\langle 5, -5, 5 \rangle| = 5\sqrt{3}$$

$$|\langle 2, 0, 0 \rangle| = 2$$

$$\cos \theta = \frac{10}{2 \cdot 5\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$$

6. (7 pts) Give an equation for the plane that is perpendicular to the line $x = t, y = 2t + 1, z = -3t - 12$ and passes through the point $(2, 5, -18)$.

$$\vec{N} = \langle 1, 2, -3 \rangle$$

$$P = (2, 5, -18)$$

$$(x-2) + 2(y-5) - 3(z+18) = 0$$

$$x + 2y - 3z = 66$$

7. (4 pts) True or false. You do not need to justify your answers. \mathbf{a} , \mathbf{b} and \mathbf{c} are vectors.

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

T

(b) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

F

(c) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|$

F

(d) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

T

8. (a) (6 pts) Find a point that lies on both the planes $P_1 : x + y + z = 1$ and $P_2 : -\frac{1}{2}x - \frac{1}{3}y + z = 1$.

$$\begin{array}{r} y+z=1 \\ -\frac{1}{3}y+z=1 \\ \hline 4z=4 \\ z=1 \\ y=0 \end{array} \quad (0, 0, 1)$$

- (b) (4 pts) Find the normal vectors for the planes P_1 and P_2 .

$$\begin{aligned} \vec{N}_1 &= \langle 1, 1, 1 \rangle \\ \vec{N}_2 &= \langle -\frac{1}{2}, -\frac{1}{3}, 1 \rangle \end{aligned}$$

- (c) (6 pts) Find a vector which is perpendicular to the normal vectors of P_1 and P_2 .

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -\frac{1}{2} & -\frac{1}{3} & 1 \end{vmatrix} = \left\langle \frac{4}{3}, -\frac{3}{2}, \frac{1}{6} \right\rangle$$

(d) (4 pts) Give the symmetric equations for the line of intersection between the planes P_1 and P_2 .

$$x = \frac{4}{3}t \quad y = -\frac{3}{2}t \quad z = \frac{1}{6}t + 1$$

$$\frac{3x}{4} = -\frac{2y}{3} = 6(z-1)$$

9. (6 pts) Find the volume of the parallelepiped defined by the vectors $\langle 3, 1, 1 \rangle$, $\langle 2, 4, 2 \rangle$ and $\langle 3, 3, 5 \rangle$.

$$\begin{vmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 3 & 5 \end{vmatrix} = 3 \begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix}$$

$$= 3(14) - 1(4) + 1(-6)$$

$$= 42 - 10 = \boxed{32}$$

10. (9 pts) If the position of particle in motion is given by the formula $\mathbf{r}(t) = \langle 5t, \cos t, \sin t \rangle$, find \mathbf{a}_T and \mathbf{a}_N , the tangential and normal components of acceleration.

$$\dot{\mathbf{r}}(t) = \langle 5t, \cos t, \sin t \rangle$$

$$\vec{v}(t) = \langle 5, -\sin t, \cos t \rangle$$

$$v(t) = \sqrt{5^2(\sin^2 t) + \cos^2 t} = \sqrt{26}$$

$$\vec{T}(t) = \left\langle \frac{5}{\sqrt{26}}, \frac{-\sin t}{\sqrt{26}}, \frac{\cos t}{\sqrt{26}} \right\rangle$$

$$\vec{a}(t) = v' \vec{T} + v \vec{T}' = 0 \left\langle \frac{5}{\sqrt{26}}, \frac{-\sin t}{\sqrt{26}}, \frac{\cos t}{\sqrt{26}} \right\rangle + \sqrt{26} \left\langle 0, \frac{-\cos t}{\sqrt{26}}, \frac{-\sin t}{\sqrt{26}} \right\rangle$$

$$\vec{a}_T = \langle 0, 0, 0 \rangle \quad \vec{a}_N = \langle 0, -\cos t, -\sin t \rangle$$

11. Consider the cone $z^2 = 3x^2 + 3y^2$.

(a) (4 pts) Convert the equation into cylindrical coordinates.

$$z^2 = 3r^2$$

(b) (4 pts) Convert the equation into spherical coordinates.

$$\frac{1}{3} = \frac{z^2}{r^2} \frac{r^2}{z^2}$$

~~$$\frac{1}{3} = \frac{z}{r}$$~~

$$\frac{1}{\sqrt{3}} = \frac{r}{z} = \tan \varphi$$

$$\varphi = \arctan\left(\frac{1}{\sqrt{3}}\right)$$

7 $\tan \pi/6$ ~~$\frac{1}{\sqrt{3}}$~~ $\frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

12. (7 pts) Calculate the integral $\int_0^{3\pi} \langle 1, \cos t, t^{-1/2} \rangle dt$.

$$\begin{aligned} \int_0^{3\pi} \langle 1, \cos t, t^{-1/2} \rangle dt &= \langle t, \sin t, 2t^{1/2} \rangle \Big|_0^{3\pi} \\ &= \langle 3\pi, 0, 2\sqrt{3\pi} \rangle - \langle 0, 0, 0 \rangle \\ &= \langle 3\pi, 0, 2\sqrt{3\pi} \rangle \end{aligned}$$

13. Identify the following surfaces.

(a) (4 pts) $\frac{x^2}{25} - \frac{y^2}{4} - \frac{z^2}{9} = 1$

hyperboloid of 2 sheets

(b) (4 pts) $\frac{x^2}{4} + \frac{y^2}{5} = z^2$

elliptic cone

(c) (4 pts) $\frac{x^2}{12} - y^2 = \frac{z}{4}$

hyperbolic paraboloid

Exam 2

Math 241

July 3, 2007

Name:

KEY

Instructions:

1. You have 90 minutes to finish the test. It should take around one hour. The exam is worth 100 points.
2. There are no calculators allowed on the test.
3. Answers involving inverse trigonometric functions are acceptable, especially if the answer does not simplify to a reference angle.
4. No ipods or music systems are allowed during the test.
5. No cell phones are allowed during the test.

1. (6 pts) What are the two different geometric interpretations of the gradient?

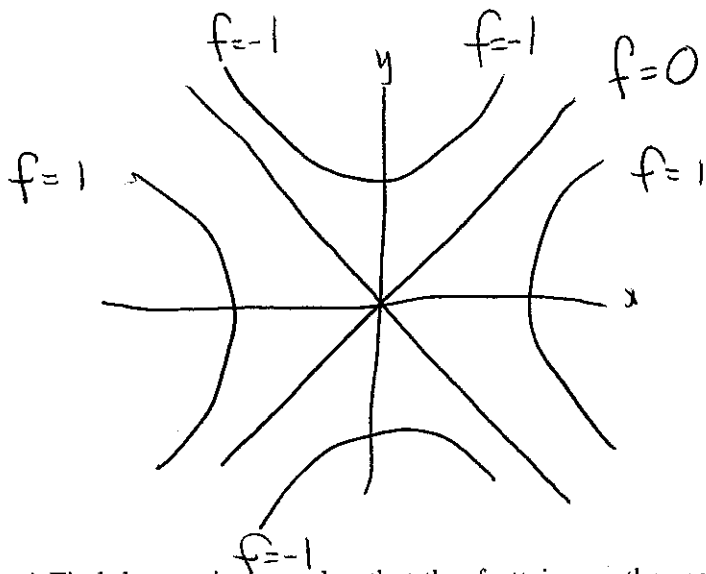
- Direction of max change inf
- Normal to level curve / surface

2. (10 pts) The temperature at a point (x, y) is given by the function $T(x, y) = 2xy - \frac{1}{2}x^2 - y$. A bug is crawling along the path traced by the curve $\mathbf{r}(t) = \langle \sqrt{1+t}, 2 + \frac{1}{3}t \rangle$. How fast is the temperature changing after the bug has been crawling for 3 seconds (time is in seconds, and the temperature is in degrees Celsius)?

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} \\ &= (2y - x) \left(\frac{1}{2} (1+t)^{-1/2} \right) + (2x - 1) \left(\frac{1}{3} \right) \\ &= \left(4 + \frac{2}{3}t - \sqrt{1+t} \right) \left(\frac{1}{2\sqrt{1+t}} \right) + \frac{(2\sqrt{1+t} - 1)}{3}\end{aligned}$$

$$\begin{aligned}\frac{dT}{dt} & \\ T'(3) &= \left(4 + \frac{2}{3} \cdot 3 - \sqrt{4} \right) \frac{1}{2\sqrt{4}} + \frac{2\sqrt{4} - 1}{3} \\ &= 1 + \frac{4-1}{3} = 1 + 1 = \boxed{2} \quad \frac{10}{15}\end{aligned}$$

3. (a) (6 pts) Draw some typical level curves with both positive and negative values of the function $f(x, y) = x^2 - y^2$ and label the value of the function on those curves.



- (b) (10 pts) Find the maximum value that the f attains on the region of the plane bounded by the circle $x^2 + y^2 = 16$.

Check critical points:

$$f_x = 2x$$

$$f_y = -2y$$

CP at $(0, 0)$. $f(0, 0) = 0$

Lagrange multipliers on the boundary:

$$f(x, y) = x^2 - y^2$$

$$g = x^2 + y^2 - 16$$

$$2x = \lambda 2x \Rightarrow x = \lambda x$$

$$2y = -\lambda 2y \Rightarrow y = \lambda y$$

$$\Rightarrow x \text{ or } y \text{ is } 0$$

Check $(0, \pm 4)$ $(\pm 4, 0)$

$$f(0, \pm 4) = -16$$

$$f(\pm 4, 0) = 16 \leftarrow \text{maximum.}$$

4. (10 pts) Determine if the following limit exists. If it does, evaluate it.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Approach $(0,0)$ along $y=mx$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m^2}{1+m^2}$$

The limit depends on the choice of m ,

So

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ doesn't exist.}$$

5. (a) (4pts) Define what it means for a function $f(x, y)$ to be continuous at the point (a, b) .

$f(x, y)$ is continuous at (a, b)

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$$

- (b) (10 pts) Show that the function $f(x, y) = x^2 + y^2$ is continuous at the point $(0, 0)$.

$$\lim_{(x, y) \rightarrow (0, 0)} x^2 + y^2 = \lim_{r \rightarrow 0} r^2 = 0$$

$$f(0, 0) = 0.$$

Thus

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0), \text{ so } f \text{ is continuous.}$$

6. (12 pts) Find and classify all critical points of the function $f(x, y) = \frac{1}{4}x^4 - 2x^2 + 2y^3 + 3y^2$.

$$f_x = x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$$

$$f_y = 6y^2 + 6y = 6y(y+1)$$

$$f_{xx} = 3x^2 - 4$$

$$f_{yy} = 12y + 6$$

$$f_{xy} = 0$$

CP	f_{xx}	f_{yy}	Δ	
(0,0)	-4	6	-24	Saddle
(0,-1)	-4	-6	24	Max
(2,0)	8	6	48	Min
(2,-1)	8	-6	-48	Saddle
(-2,0)	8	6	48	Min
(-2,-1)	8	-6	-48	Saddle

7. Consider the function $f(x, y) = x^2 + y^3 - xy$.

(a) (4 pts) Find $\nabla f(x, y)$.

$$\nabla f(x, y) = \langle 2x - y, 3y^2 - x \rangle$$

(b) (4 pts) Find the direction of the maximum directional derivative for the function $f(x, y) = x^2 + y^3 - xy$ at the point $(2, -1)$.

$$\nabla f(2, -1) = \langle 4 + 1, 12 + 1 \rangle = \langle 5, 13 \rangle$$

(c) (6 pts) Find the directional derivative of f in the direction of the vector $\langle 1, 2 \rangle$ at the point $(2, -1)$.

$$\nabla f \cdot \vec{v} \quad \vec{v} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$D_{\vec{v}} f(2, -1) = \langle 5, 13 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{5 + 26}{\sqrt{5}} = \frac{31}{\sqrt{5}}$$

8. (a) (9 pts) Find the equation of the plane tangent to the surface $z = x^2 + y^2$ at the point $(2, 2, 8)$.

$$z = f(x, y) = x^2 + y^2$$

$$f_x = 2x$$

$$f_y = 2y$$

$$f_x(2, 2) = 4$$

$$f_y(2, 2) = 4$$

$$z - 8 = 4(x - 2) + 4(y - 2)$$

$$z - 8 = 4x - 8 + 4y - 8$$

$$z = 4x + 4y - 8$$

- (b) (9 pts) Now give a good estimate of the z -coordinate of the point $(2.1, 1.9, z)$ that lies on the surface.

~~$f(2.1, 1.9)$~~

$$f(2.1, 1.9) \approx f(2, 2) + f_x(2, 2)(.1) + f_y(2, 2)(-.1)$$

$$\approx 8 + 4(.1) + 4(-.1) = 8$$

Exam 3

Math 241

July 17, 2007

Name:

KEY

Instructions:

1. You have 90 minutes to finish the test. It should take around one hour. The exam is worth 100 points.
2. You must show all work to receive full credit.
3. There are no calculators allowed on the test.
4. Answers involving inverse trigonometric functions are acceptable, especially if the answer does not simplify to a reference angle.
5. No ipods or music systems are allowed during the test.
6. No cell phones are allowed during the test.

1. Set up the integral of the function $f(x, y) = xy^2$ over the region R which is bounded by the circle of radius 2 centered at the origin using:

(a) (5 points) rectangular coordinates;

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xy^2 dy dx$$

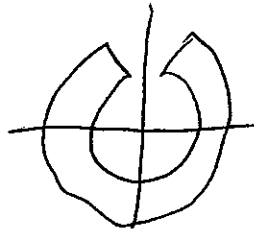
(b) (5 points) polar coordinates.

$$\int_0^{2\pi} \int_0^2 (r \cos \theta) (r \sin \theta)^2 r dr d\theta$$

(c) (8 points) Now evaluate the integral in polar coordinates.

$$\begin{aligned} \int_0^{2\pi} \int_0^2 r^4 \sin^2 \theta \cos \theta dr d\theta &= \int_0^{2\pi} \left[\frac{1}{5} r^5 \sin^2 \theta \cos \theta \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{32}{5} \sin^2 \theta \cos \theta d\theta = \frac{32}{5} \cdot \frac{1}{3} \sin^3 \theta \Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

2. (5 points) Draw an example of a region R in the xy -plane which is neither x - nor y -simple.



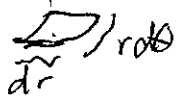
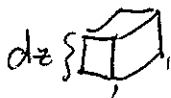
3. (10 points) Use a double integral to find the volume in the first octant which lies beneath the plane $8x + 4y + 2z = 16$.

$$\begin{aligned}
 \int_0^2 \int_0^{4-2x} (8 - 2y - 4x) \, dy \, dx &= \int_0^2 \left[8y - y^2 - 4xy \right]_0^{4-2x} \, dx \\
 &= \int_0^2 (32 - 16x) - (16 - 16x + 4x^2) - (16x - 8x^2) \, dx \\
 &= \int_0^2 4x^2 - 16x + 16 \, dx = \left[\frac{4}{3}x^3 - 8x^2 + 16x \right]_0^2 \\
 &= \frac{32}{3} - 32 + 32 = \frac{32}{3}
 \end{aligned}$$

4. (a) (4 points) State the form of the differential dV for cylindrical coordinates.

$$dV = r \, dz \, dr \, d\theta$$

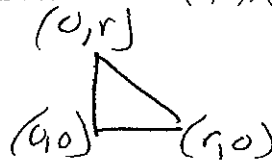
- (b) (6 points) Justify the cylindrical expression of the differential dV by explaining the subdivision for the appropriate Riemann Sum.



The width of the projection onto the xy -plane is given by ~~$r \, d\theta$~~ $r \, d\theta$. It depends on how far you are from the origin.

5. (a) (10 points) Find the centroid of the triangular region with vertices $(0, 0)$, $(0, r)$, and $(r, 0)$ which has uniform density $\delta(x, y) = 1$.

$$A = \frac{1}{2} r^2$$



$\bar{x} = \bar{y}$ by symmetry

$$\bar{x} = \frac{1}{A} \iint_R x \, dx \, dy = \frac{2}{r^2} \int_0^r \int_0^{r-x} x \, dy \, dx$$

$$= \frac{2}{r^2} \int_0^r x y \Big|_0^{r-x} dx = \frac{2}{r^2} \int_0^r (rx - x^2) dx = \frac{2}{r^2} \left(\frac{r}{2} x^2 - \frac{1}{3} x^3 \right) \Big|_0^r$$

$$= \frac{2}{r^2} \left(\frac{r^3}{2} - \frac{r^3}{3} \right) = \frac{2}{r^2} \left(\frac{r^3}{6} \right) = \frac{r}{3}$$

Centroid: $\left(\frac{r}{3}, \frac{r}{3} \right)$

- (b) (7 points) Use the first theorem of Pappus to verify that the volume of a right circular cone with radius r and height r is given by $\frac{1}{3}\pi r^3$

$$V = A \cdot d$$

$$= \frac{1}{2} r^2 \cdot 2\pi \left(\frac{r}{3}\right) = \frac{1}{3} \pi r^3$$

6. (10 points) Find the mass of the solid with density $\delta(x, y, z) = z^3$ bounded by the sphere $x^2 + y^2 + z^2 = 25$ and the cone $z = \sqrt{3(x^2 + y^2)}$. (Hint: Use spherical coordinates.)

$$\iiint_{\mathcal{R}} z^3 dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^5 (\rho \cos \varphi)^3 \rho^2 \sin \varphi d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \left[\frac{1}{6} \rho^6 \cos^3 \varphi \sin \varphi \right]_0^5 d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi/6} \frac{5^6}{6} \cos^3 \varphi \sin \varphi d\varphi d\theta$$

$$= \int_0^{2\pi} \left[\frac{5^6}{6 \cdot 4} \cos^4 \varphi \right]_0^{\pi/6} d\theta = \int_0^{2\pi} \frac{5^6}{24} \left(1 - \left(\frac{\sqrt{3}}{2}\right)^4 \right) d\theta$$

$$= \int_0^{2\pi} \frac{5^6}{24} \left(1 - \frac{9}{16} \right) d\theta = \frac{5^6}{24} \left(\frac{7}{16} \theta \right) \Big|_0^{2\pi}$$

$$= \frac{5^6 \cdot 7}{12 \cdot 16} \pi$$

$$\frac{9-16}{16} = -\frac{7}{16} \Rightarrow \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

7. (a) (6 points) Parameterize the top half of the sphere of radius 5 centered at the origin in terms of r and θ .

$$\langle r \cos \theta, r \sin \theta, \sqrt{25 - r^2} \rangle$$

- (b) (6 points) Set up, but do not compute, a double integral in terms of r and θ for the surface area of the top half of the sphere of radius 5 centered at the origin.

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \theta & \sin \theta & \frac{1}{2}(25 - r^2)^{-1/2} 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \left\langle \frac{r^2 \cos \theta}{\sqrt{25 - r^2}}, -\frac{r^2 \sin \theta}{\sqrt{25 - r^2}}, r \right\rangle$$

$$A = \int_0^{2\pi} \int_0^5 \sqrt{\frac{r^4 \cos^2 \theta}{25 - r^2} + \frac{r^4 \sin^2 \theta}{25 - r^2} + r^2} \, dr \, d\theta$$

$$\sqrt{r^2 + \left(r \left(\frac{1}{2} \frac{1}{\sqrt{25 - r^2}} \right) (-2r) \right)^2 + 0}$$

$$\sqrt{r^2 + \left(\frac{-r^2}{\sqrt{25 - r^2}} \right)^2}$$

$$= \sqrt{r^2 + \left(\frac{r^4}{25 - r^2} \right)}$$

8. (a) (4 points) Write down the general form of the Jacobian for the change of variables given by $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$.

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- (b) (5 points) Give a change of variables for x and y in terms of u and v which would assist with computing the area bounded by the curves $x = y^4$, $x = 4y^4$, $y = x^3$, $y = 15x^3$.

$$\begin{aligned} x &= u y^4 & y &= v x^3 \\ x &= u v^4 y^{12} & y &= v u^3 y^{12} \end{aligned}$$

$$\boxed{x = \left(\frac{1}{u v^4}\right)^{1/11} \quad y = \left(\frac{1}{v u^3}\right)^{1/11}}$$

- (c) (9 points) Using the change of variables $x = 3u$, $y = 4v$, find the area bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

Region bounded by $\frac{(3u)^2}{9} + \frac{(4v)^2}{16} = 1 \implies u^2 + v^2 = 1$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = 12$$

$$A = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 12 \, du \, dv = \boxed{12\pi}$$