Exam 4

Math 241

August 1, 2007

Name: KEY

Instructions:

1. You have 90 minutes to finish the test. It should take around one hour. The exam is worth 100 points.
2. You must show all work to receive full credit.
3. There are no calculators allowed on the test.
4. Answers involving inverse trigonometric functions are acceptable, especially if the answer does not simplify to a reference angle.
5. No ipods or music systems are allowed during the test.
6. No cell phones are allowed during the test.

1. Consider the vector field $\mathbf{F} = \langle xyz^2, 2xz, -zy \rangle$.

(a) (6 points) Calculate the divergence of $\mathbf{F}$ at the point $(1, 1, 2)$.

$$\nabla \cdot \vec{F} = (yz^2 + 0 + -y) = yz^2 - y$$

$$\nabla \cdot F(1,1,2) = 1(4) - 1 = \boxed{3}$$

(b) (6 points) Determine the curl of $\mathbf{F}$ at the general point (x, y, z).

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz^2 & 2xz & -zy \end{vmatrix} = \langle -z - 2x, 2xyz, 2z - xz^2 \rangle$$

2. (6 points) State one similarity between Green's theorem and Stokes' theorem.

3. Consider the vector field $\mathbf{F}(x, y) = \langle y, x \rangle$

(a) (6 points) Show that the vector field $\mathbf{F}(x, y) = \langle y, x \rangle$ is conservative using a theorem stated in class.

$$\frac{\partial Q}{\partial x} = 1 = \frac{\partial P}{\partial y}$$

(b) (8 points) Find a potential function f for $\mathbf{F}$.

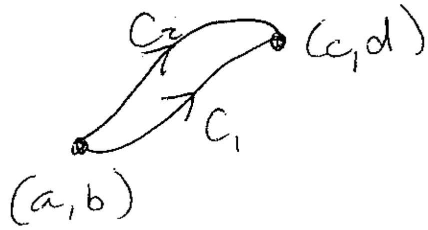
$$\int y \, dx = \cancel{\int} \; yx + \varphi(x)$$

$$\int x \, dy = xy + \theta(y) \qquad \varphi(x) = \theta(y) = 0$$

$$f(x, y) = xy$$

4. Let the vector field $\mathbf{F} = \langle P, Q \rangle$ have the property that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ at each point in the xy-plane.

(a) (4 points) Let C_1 and C_2 be two distinct, nonintersecting, smooth paths from the point (a, b) to the point (c, d). Define a closed path C in terms of C_1 and C_2 which starts and ends at (a, b).



$$C = C_1 + C_2^-$$

(b) (9 points) Use Green's Theorem to show that $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{T} ds = 0$.

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0 \, dA = 0$$

(c) (9 points) Now show that $\displaystyle\int_{C_1} \mathbf{F} \cdot \mathbf{T} ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T} ds$.

$$0 = \oint_C \vec{F} \cdot \vec{T}\, ds = \int_{C_1 + C_2^-} \vec{F} \cdot \vec{T}\, ds = \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2^-} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds - \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$

5. (10 points) Evaluate the surface integral $\int\int_S x^2y\,dS$ where S is the part of the plane $z = 1+2x+3y$ that lies above the rectangle defined by $0 \le x \le 3$ and $0 \le y \le 2$.

$S: \vec{r}(x,y) = \langle x, y, 1+2x+3y \rangle$

$$\iint_S x^2y\,dS$$

$$= \iint x^2y \sqrt{\left(\frac{\partial(y,z)}{\partial(x,y)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(x,y)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(x,y)}\right)^2}$$

$$= \int_0^2\int_0^3 x^2y \sqrt{\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix}^2 + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2}$$

$$= \int_0^2\int_0^3 x^2y\sqrt{4+9+1}\,dx\,dy$$

$$= \sqrt{14}\int_0^2\int_0^3 x^2y\,dx\,dy = \sqrt{14}\int_0^2 \frac{x^3}{3}y\Big]_0^3\,dy$$

$$= \sqrt{14}\int_0^2 9y\,dy = 9\sqrt{14}\left(\frac{1}{2}y^2\right]_0^2$$

$$= 18\sqrt{14}$$

6. Consider the vector field $\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ and the surface S which is the cube defined by the three coordinate planes and the planes $x = 3$, $y = 3$ and $z = 3$.

(a) (4 points) How many surface integrals are necessary to compute the flux of $\mathbf{F}$ across S directly?

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(b) (10 points) Use the divergence theorem to compute the flux of the vector field $\mathbf{F}$ across the surface S.

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_R \vec{\nabla} \cdot \vec{F}\, dV$$

$$= \int_0^3 \int_0^3 \int_0^3 (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_0^3 \int_0^3 x^2 + (2y + 2z)x \Big]_0^3\, dy\, dz = \int_0^3 \int_0^3 (9 + 6y + 6z)\, dy\, dz$$

$$= \int_0^3 3y^2 + (9 + 6z)y \Big]_0^3\, dz = \int_0^3 27 + 27 + 18z\, dz$$

$$= 54z + 9z^2 \Big]_0^3 = 162 + 81 = \boxed{243}$$

7. (10 points) If the density of a wire that lies along the curve $x = 3t, y = t^4,\ 0 \le t \le 1$, is given by the function $f(x, y) = xy$, find the total mass of the wire.

$$\vec{r}(t) = \langle 3t, t^4 \rangle$$

$$\int_C xy\,ds = \int_C 3t(t^4)\sqrt{9+(4t^3)^2}\,dt$$

$$= \int_0^1 3t^5\sqrt{9+16t^6}\,dt = \frac{2}{3}\cdot\frac{3}{6\cdot 16}(9+16t^6)^{3/2}\Big]_0^1$$

$$= \frac{1}{48}(9+16t^6)^{3/2}\Big]_0^1 = \frac{1}{48}(25)^{3/2} - \frac{1}{48}(9)^{3/2}$$

$$= \frac{125}{48} - \frac{27}{48} = \frac{98}{48} = \boxed{\frac{49}{24}}$$

8. (a) (4 points) State Stokes' theorem for a continouously differentiable vector field **F** and a piecewise smooth surface S with boundary curve C.

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_S (\text{curl } \vec{F}) \cdot \vec{n}\, dS$$

(b) (8 points) Set up **but do not compute** the surface integral equivalent to the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$$

where $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise when viewed from above. (Your answer should be a surface integral complete with bounds of integration for your chosen parameterization.)

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0-0, -(0-0), 1+2y \rangle$$

$$\iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS = \iint 1 + 2y \; dx\,dy$$

surface: $\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 2 - r\sin\theta \rangle$

$$\int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta) \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta)\, r\, dr\,d\theta.$$