Exam 3

Math 124

April 23, 2009

Name: KEY

Instructions:

1. You have 60 minutes to complete the exam.
2. Put away all cell phones, ipods etc.
3. There are no calculators allowed.
4. You must show all work to recieve full credit.
5. Carefully read each question and follow all directions.

1	2	3	4	5	6	7	8	9

1. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix},$$

Compute the following if the operation is defined; if the operation is undefined, write "undefined."

(a) (4 pts) $A + A$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \\ 4 & 8 \end{bmatrix}$$

(b) (4 pts) B^T

$$B^T = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(c) (4 pts) $B^T + (-2)A$

$$B^T + (-2A) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ -2 & -4 \\ -4 & -8 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ -3 & -8 \end{bmatrix}$$

2. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

Perform the indicated matrix operation if it is defined; otherwise write "undefined":

(a) (4 pts) AB

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 3 & 6 & 1 \\ 6 & 12 & 2 \end{bmatrix}$$

(b) (4 pts) C^2

$$\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) (4 pts) BC

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

Undefined

3. (5 pts) Verify that following pair of matrices are inverses of one another:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \checkmark$$

4. Suppose an economy has three goods – cotton, lumber, and peaches, respectively, and it is modeled by the consumption matrix:

$$C = \begin{bmatrix} .2 & 0 & .6 \\ .6 & .1 & 0 \\ .2 & .4 & .3 \end{bmatrix}$$

(a) (3 pts) Which industries, if any, are profitable? You must mathematically support your answer.

Cotton: $.2 + .6 + .2 = 1$ Not profitable

Lumber: $0 + .1 + .4 = .5 < 1$ Profitable

Peaches: $.6 + 0 + .3 = .9 < 1$ Profitable

(b) (3 pts) Is the economy productive? You must mathematically support your answer.

Row sums: R_1 $.2 + 0 + .6 = .8 < 1$

R_2 $.6 + .1 + 0 = .7 < 1$

$.2 + .4 + .3 = .9 < 1$

All less than 1, so it is productive.

5. Jack and his wife Jill produce pails and water. Every \$1 pail produced uses \$0.80 pail and \$0.40 water, while every \$1 water produced uses \$0.30 pail and \$0.30 water. Suppose that Jack and Jill have recieved orders for \$840 pail and \$980 water. Find a production schedule using the Leontief Open Model.

(a) (3 pts) Find the consumption matrix C.

$$C = \begin{bmatrix} .8 & .3 \\ .4 & .3 \end{bmatrix}$$

(b) (3 pts) Compute $I - C$.

$$I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .8 & .3 \\ .4 & .3 \end{bmatrix} = \begin{bmatrix} .2 & -.3 \\ -.4 & .7 \end{bmatrix}$$

(c) (3 pts) Find the inverse $(I - C)^{-1}$.

$$\left[\begin{array}{cc|cc} .2 & -.3 & 1 & 0 \\ -.4 & .7 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & -3 & 10 & 0 \\ -4 & 7 & 0 & 10 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -3/2 & 5 & 0 \\ -4 & 7 & 0 & 10 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -3/2 & 5 & 0 \\ 0 & 1 & 20 & 10 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 35 & 15 \\ 0 & 1 & 20 & 10 \end{array}\right]$$

$$(I - C)^{-1} = \begin{bmatrix} 35 & 15 \\ 20 & 10 \end{bmatrix}$$

(d) (3 pts) Determine the production vector for the given external demand.

$$\begin{bmatrix} 35 & 15 \\ 20 & 10 \end{bmatrix} \begin{bmatrix} 840 \\ 980 \end{bmatrix} = \begin{bmatrix} 35 \cdot 840 + 15 \cdot 980 \\ 20 \cdot 840 + 10 \cdot 980 \end{bmatrix} = \Bigg[\begin{matrix} 29400 + \cancel{14700} \\ \end{matrix}$$

$35 \cdot 840 = 25200 + 4200$

$= 29400$

$9800 + 4900$

(e) (3 pts) What dollar amount must each industry produce to meet the external demand?

\$ $(35 \cdot 840 + 15 \cdot 980)$ of Pail

\$ $(20 \cdot 840 + 10 \cdot 980)$ of Water

6. Consider the system

$$\begin{aligned} 8x_1 + 3x_2 &= 4 \\ 5x_1 + 2x_2 &= -2 \end{aligned}$$

(a) (3 pts) Write the system as a matrix equation.

$$\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

(b) (3 pts) Find the inverse of the coefficient matrix for the system.

$$\left[\begin{array}{cc|cc} 8 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$$

$$\left.\begin{array}{cc|cc} 1 & 3/8 & 1/8 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right.$$

$$\left.\begin{array}{cc|cc} 1 & 3/8 & 1/8 & 0 \\ 0 & 1/8 & -5/8 & 1 \end{array}\right.$$

$$\left.\begin{array}{cc|cc} 1 & 3/8 & 1/8 & 0 \\ 0 & 1 & -5 & 8 \end{array}\right.$$

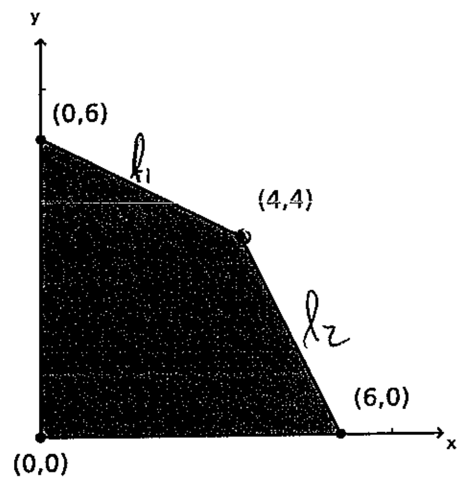
$$\left.\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -5 & 8 \end{array}\right.$$

$$\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$$

(c) (4 pts) Use the inverse of the coefficient matrix to solve the system.

$$\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 8+6 \\ -20-16 \end{bmatrix} = \begin{bmatrix} 14 \\ -36 \end{bmatrix}$$

7. (5 pts) Determine the values for a so that the maximum value of $P = a \cdot x + 12y$ on the feasibility region below occurs at $(4, 4)$.



l_1 has slope $\frac{-2}{4} = -\frac{1}{2}$

l_2 has slope $\frac{-4}{2} = -2$

Lines of constancy have slope $y = \frac{-a}{12}x + \frac{P}{12}$

$\frac{-a}{12}$

Need

$$-2 < \frac{-a}{12} < -\frac{1}{2}$$

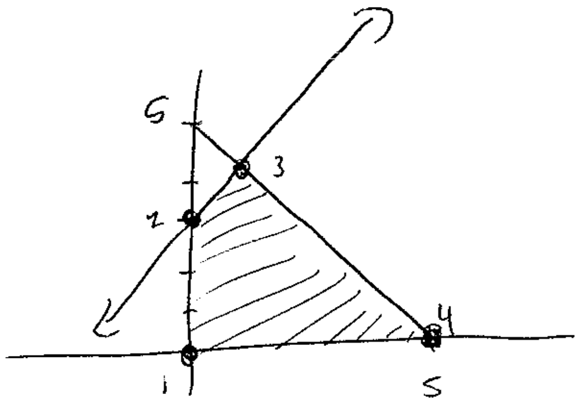
$$-24 < -a < -6$$

$$\boxed{6 < a < 24}$$

8. (15 pts) Solve the linear program

$$\begin{array}{lrcl} \text{Minimize} & z & = & 2x - y \\ \text{subject to} & (1)\ x - y & \geq & -3 \\ & (2)\ x + y & \leq & 5 \\ & x & \geq & 0 \\ & y & \geq & 0. \end{array}$$

$\Rightarrow \quad y \leq x+3$



PT 1: (0,0)

PT 2: (0,3)

PT 3: $y = x+3 = 5-x$

$2x = 2$

$x = 1$ $\quad y = 4$

(1,4)

PT 4: (5,0)

PT	$2x-y$
(0,0)	0
(0,3)	-3 ← Minimum
(1,4)	-2
(5,0)	10

8

$\boxed{\text{Min } -3 \text{ @ } (0,3)}$

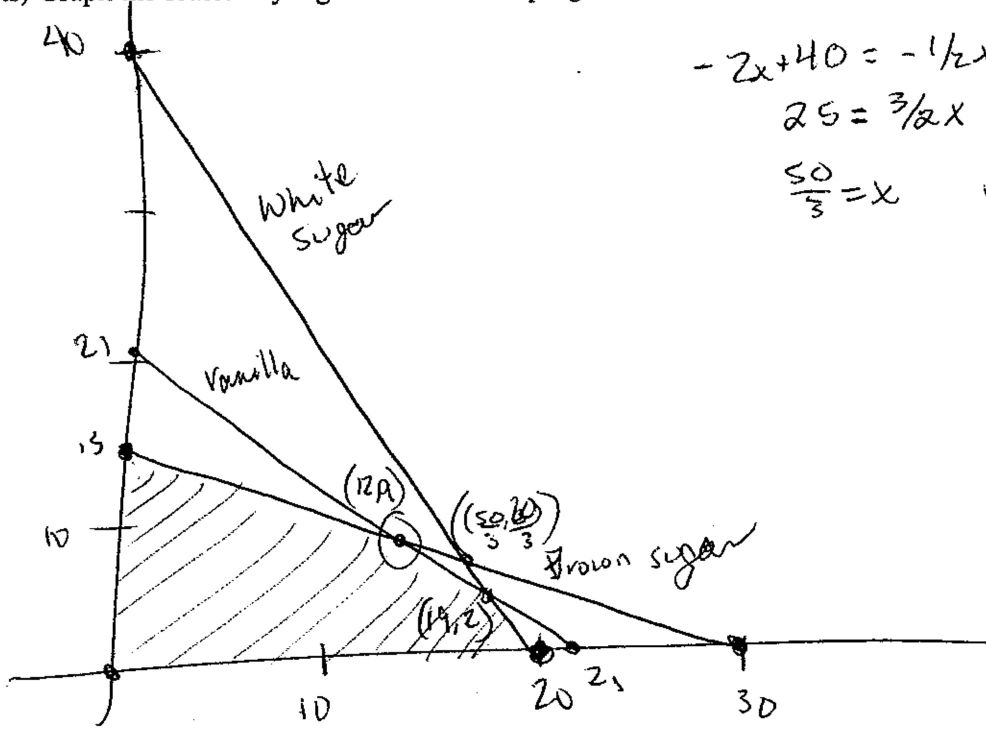
9. You own a bakery and you make chocolate chip cookies and molasses cookies. The chocolate chip cookies use 2 cups of white sugar, one cup of brown sugar and a tablespoon of vanilla. The molasses cookies use one cup of white sugar, two cups of brown sugar and a tablespoon of vanilla. In your bakery you have 40 cups of white sugar, 30 cups of brown sugar and 21 tablespoons of vanilla. You sell chocolate chip cookies for \$1 each and the molasses cookies for \$1.25 each.

(a) (5 pts) Set up the linear program which maximizes the revenue of your bakery.

Maximize $x + 1.25y$

Subject to
$2x + y \leq 40 \quad \rightarrow \quad y \leq -2x + 40$
$x + 2y \leq 30 \quad \rightarrow \quad y \leq -\frac{1}{2}x + 15$
$x + y \leq 21 \quad\quad y \leq -x + 21$
$x \geq 0$
$y \geq 0$

(b) (5 pts) Graph the feasibility region of the linear program.



$-2x + 40 = -\frac{1}{2}x + 15$
$25 = \frac{3}{2}x$
$\frac{50}{3} = x \quad y = \frac{20}{3}$

(c) (5 pts) Find the optimal solution to the linear program.

$-x+21=-2x+40$ Vanilla + White sugar

$x=19,\ y=2$ (19, 2)

$-x+21=-\frac{1}{2}x+15$ Vanilla + Brown sugar

$6=\frac{1}{2}x$

$12=x,\ y=9$ (12, 9)

PT	x + 1.25y
(0, 0)	0
(0, 15)	18.75
(12, 9)	12 + 11.25 = 23.25 ← Max
(19, 2)	19 + 2.50 = 21.50
(20, 0)	20

(d) (5 pts) Determine which resources are exhausted and which are not in this problem.

Brown sugar + vanilla are exhausted.

White suger is <u>not</u> exhausted.