Further Results on Bar k-Visibility Graphs

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Abstract

A bar visibility representation of a graph G is a collection of horizontal bars in the plane corresponding to the vertices of G such that two vertices are adjacent if and only if the corresponding bars can be joined by an unobstructed vertical line segment. In a bar k-visibility graph, two vertices are adjacent if and only if the corresponding bars can be joined by a vertical line segment that intersects at most k other bars. Bar k-visibility graphs were introduced by Dean, Evans, Gethner, Laison, Safari, and Trotter in [3],[4]. In this paper, we present sharp upper bounds on the maximum number of edges in a bar k-visibility graph on n vertices and the largest order of a complete bar k-visibility graph. We also discuss regular bar k-visibility graphs and forbidden induced subgraphs of bar k-visibility graphs.

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1 Introduction

The idea of representing a graph using a visibility relation has received much attention due to its applications to circuit layout (see [9], additional references in [1]). Define a *bar visibility representation* of a graph G to be a set of disjoint horizontal closed line segments (or *bars*) in the plane in one-to-one correspondence with the vertices of G such that v and w are adjacent in G if and only if a vertical line segment can be drawn joining their associated bars that does not intersect any other bar. We say that G is a bar visibility graph if it has a bar visibility representation. (In the literature, these graphs have also been referred to as "visibility graphs" or "strong visibility graphs.") This particular model was first introduced by Luccio et al. [6]. They observed that bar visibility graphs must be planar, and later

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provided a characterization of bar visibility graphs with the restriction that bars have distinct x coordinates as endpoints [7]. Later, Tamassia and Tollis [10] gave a somewhat complicated characterization for the general case and obtained some results concerning connectivity and bar visibility representations. Andreae [1] showed that determining whether a given graph is a bar visibility graph is an NP-complete problem.

Recently, Dean, Evans, Gethner, Laison, Safari, and Trotter introduced in [4] the following generalization of bar visibility graphs. A *bar k-visibility representation* of a graph G is a bar representation in which v and w are adjacent in G if and only if a vertical line segment can be drawn joining their associated bars that intersects *at most* k other bars. We denote the family of bar k-visibility graphs as $\mathcal{F}_k$. Notice that $\mathcal{F}_0$ is the family of bar visibility graphs defined above. Dean et al. obtain a bound on the number of edges for any graph in $\mathcal{F}_k$: If $n(G) \geq 2k+2$, then G has at most $(k+1)(3n - \frac{7}{2}k - 5) - 1$ edges. They give a construction showing that the complete graph on $4k+4$ vertices K_{4k+4} is in $\mathcal{F}_k$ and conjecture an improved edge bound of $(k+1)(3n-4k-6)$, which is attained by K_{4k+4}. They prove this conjecture for $k = 0, 1$, and use their edge bound to establish $K_{5k+5} \notin \mathcal{F}_k$. Other results include bounds on the thickness and the chromatic number of bar k-visibility graphs.

In this paper, we prove that the upper bound on the number of edges conjectured in [4] is correct, yielding K_{4k+4} as the largest complete bar k-visibility graph. We also prove that for each k, $\mathcal{F}_{k-1}$ and $\mathcal{F}_k$ are incomparable under set inclusion. We restrict the graphs that are in $\mathcal{F}_k$ for some k by proving that triangle-free graphs which are nonplanar are forbidden as induced subgraphs in bar k-visibility graphs. Finally, inspired by the result that the only regular interval graphs are complete graphs, we prove that if G is regular of degree $d < 2k+2$ and $G \in \mathcal{F}_k$, then G is a complete graph. However, we have constructions of $(2k+2)$-regular non-complete graphs with bar k-visibility representations for $k \in \{0, 1, 2, 3, 4\}$.

Bar k-visibility graphs can be seen as a generalization of interval graphs. In particular, as k approaches infinity, it is easy to see that $\mathcal{F}_k$ approaches the family of interval graphs; hence results on interval graphs inspired many of our investigations into bar k-visibility graphs. A variation of interval graphs that has been studied is the idea of t-interval graphs, where each vertex of G is allotted t distinct intervals in its interval representation. This idea was extended to bar visibility graphs in [2], where each vertex of G is permitted t bars in its representation, and vertices are adjacent if there is a direct line of sight between any of their t bars. Another well-studied variation of bar visibility graphs is that ϵ-visibility graphs, introduced by Melnikov [8]. These graphs are defined just as bar k-visibility graphs, except bars are replaced with arbitrary intervals which may not contain their endpoints. Wismath [11] and, independently, Tamassia and Tollis [10] gave a very simple characterization of ϵ-visibility graphs.

Throughout this paper, all graphs are simple graphs with no loops. A *bar* refers to a closed interval of the real line with an associated height. A *bar k-representation* of a graph G, $k \geq 0$, is a one-to-one correspondence between the vertices of G and a set of bars such that vertices of G are adjacent if and only if there is a vertical line segment joining their associated bars that intersects at most k other bars. (Note that in contrast to some other visibility models, a sight line of zero width is sufficient in this model.) When we are referring to a particular bar k-visibility representation of a graph G, $B(v)$ will refer to the bar associated with v and $I(v)$ will refer to the projection of $B(v)$ onto the x-axis. We also use $u \leftrightarrow v$ to denote that u and v are adjacent and $u \not\leftrightarrow v$ to denote that u and v are nonadjacent.

2 An Upper Bound on the Number of Edges

Dean et al. [4] gave an upper bound of $(k+1)(3n - \frac{7}{2}k - 5) - 1$ on the number of edges in an n-vertex bar k-visibility graph with $n \geq 2k+2$. They also conjectured an upper bound of $(k+1)(3n-4k-6)$, which would be sharp by a construction given in [4]. We prove their conjectured upper bound on the number of edges by refining their edge-counting technique.

Theorem 1. *If G is a bar k-visibility graph with more than $2k+2$ vertices, G has at most $(k+1)(3n-4k-6)$ edges.*

Proof. Consider a bar k-visibility representation of G with vertices $v_1, v_2, \ldots, v_n$. We may assume that no two bars are at the same height, and hence we index the bars in order such that v_1 is the topmost bar and v_n is the bottommost bar. As noted in [4], we may also assume that the left and right endpoints of the associated intervals are distinct. If the endpoints are not distinct, then perturbing the endpoints slightly cannot decrease the number of edges in the resulting graph.

We sweep a vertical line from left to right over the representation, counting the number of edges that are created as we encounter sightlines. When the left endpoint of a bar $B(v)$ is encountered, the number of visibility-blocking bars is only increased. Hence the only new visibilities involve the new bar. If the left endpoint of $B(v)$ is the i^{th} left endpoint encountered, then when $i \leq 2k+2$, $B(v)$ can see at most $i-1$ other bars. When $i > 2k+2$, $B(v)$ can possibly see $k+1$ bars above and $k+1$ bars below, for a total of $2k+2$ new edges. Thus the maximum number of edges counted by encountering left endpoints is

$$\begin{aligned}\sum_{i=1}^{2k+2}(i-1) + \sum_{i=2k+3}^{n}(2k+2) &= \frac{(2k+2)(2k+1)}{2} + (n-2k-2)(2k+2) \\ &= (k+1)(2n-2k-3).\end{aligned}$$

When the right endpoint of a bar $B(w)$ is encountered, the number of visibility-blocking bars between other bars is decreased and new visibilities may be created. If the right endpoint of $B(w)$ is the i^{th} right endpoint encountered, then when $i \leq n-2k-2$, up to $k+1$ bars above $B(w)$ have the potential to each see one new bar below $B(w)$. Hence, at most $k+1$ new visibilities may be created. Once there are only $2k+2$ bars remaining, each time a bar ends the potential number of new edges decreases by one. Hence when $n-2k-2 < i < n-k-1$, there are at most $n-k-1-i$ new visibilities created. When $i \geq n-k-1$, no new sightlines are created since every bar has already seen every other remaining bar. Thus the maximum number of edges counted by encountering right endpoints is

$$\sum_{i=1}^{n-2k-2}(k+1)+\sum_{i=n-2k-1}^{n-k-2}(n-k-1-i)+\sum_{i=n-k-1}^{n}0=(n-2k-2)(k+1)+\frac{k(k+1)}{2}$$
$$=(k+1)(n-\frac{3}{2}k-2).$$

Hence as was shown in [4], we have an upper bound of $(k+1)(3n-(7/2)k-5)$ for the number of edges in G.

Notice, however, that this bound is attained only if the top $k+1$ and the bottom $k+1$ bars are among the first $2k+2$ left endpoints (as we must begin a bar with at least $k+1$ bars above and $k+1$ bars below as soon as we are able to do so) and the last $2k+2$ right endpoints (as we must also end bars with $k+1$ bars above and $k+1$ bars below as long as we are able to do so). This however implies that we have twice counted the $(k+1)^2$ edges between these two sets of vertices, once when we encountered their left endpoints and once when bars between them ended. If, on the other hand, a bar from the top $k+1$ begins after at least $2k+2$ other bars have arrived, then we do not gain $2k+2$ new edges when it begins; when $B(v_i)$ begins, we gain at most $k+1+i-1$ new edges, $k+1$ from the bars below and $i-1$ from the bars above. Similarly, if a bar $B(v_i)$ for $i\leq k+1$ ends among the first $n-(2k-2)$, then there are not $k+1$ bars above it that can gain visibility. Instead, we gain at most $i-1$ new visibilities. The same holds for bars among the bottom $k+1$. We use these two facts to improve our edge bound.

Let ℓ be the number of bars of the top $k+1$ whose left endpoint is among the first $2k+2$ left endpoints and whose right endpoint is among the last $2k+2$ right endpoints, and m the similar number of bars in the bottom $k+1$. We observe first that each of the ℓm edges between these two sets of vertices is counted both when these bars begin (as a left-endpoint edge) and when visibility increases between them sufficiently as bars end (as a right-endpoint edge). For the remaining $k+1-\ell$ bars of the top $k+1$, they either begin among the last $n-(2k+2)$ or end among the first $n-(2k+2)$, or perhaps both. Either the left endpoint of these bars or the right endpoint of these bars, then, does not contribute the maximum possible number of edges. When $B(v_i)$ begins late or ends early for $i\leq k+1$, we overcount by $i-1$; thus we have overcounted the fewest number of edges when these $k+1-\ell$ bars are $B(v_{k+1}),B(v_k),\ldots,B(v_{k+1-\ell})$. In this case, our edge bound has overcounted at least

$$1+2+\cdots+(k+1-\ell)=\frac{1}{2}(k+2-\ell)(k+1-\ell)$$

edges. Similarly, we obtained at least an extra $\frac{1}{2}(k+2-m)(k+1-m)$ in our edge count by assuming the $k+1-m$ bars from the bottom $k+1$ yielded $2k+2$ edges when they began and $k+1$ edges when they ended. Therefore our graph has at most

$$(3n-\frac{7}{2}k-5)(k+1)-[\ell m+\frac{1}{2}(k+2-\ell)(k+1-\ell)+\frac{1}{2}(k+2-m)(k+1-m)]$$

edges; we seek to minimize the function

$$f(\ell, m) = \ell m + \frac{1}{2}(k+2-\ell)(k+1-\ell) + \frac{1}{2}(k+2-m)(k+1-m)$$
$$= \frac{1}{2}(\ell+m)^2 - \frac{2k+3}{2}(\ell+m) + (k+1)(k+2)$$

As this is a quadratic in $(\ell + m)$, we find that local extrema occur when $\ell + m = \frac{2k+3}{2}$, yielding a minimum objective value of $\frac{8k^2+24k+14}{16}$. Hence $f(\ell, m) \geq \frac{1}{2}k^2 + \frac{3}{2}k + \frac{7}{8}$, and therefore our graph has at most

$$\left(3n - \frac{7}{2}k - 5\right)(k+1) - \left(\frac{1}{2}k^2 + \frac{3}{2}k + \frac{7}{8}\right) = 3nk + 3n - 4k^2 - 10k - \frac{47}{8}$$

edges, and since the number of edges must be integer-valued, we get an upper bound of

$$3nk + 3n - 4k^2 - 10k - 6 = (k+1)(3n - 4k - 6).$$

□

Corollary 2. *If K_n is a bar k-visibility graph, then $n \leq 4k + 4$.*

Proof. If $n = 4k + 4 + m$, then K_n has

$$(4k+4+m)(4k+4+m-1)\frac{1}{2} = 8k^2 + 14k + 4mk + 6 + \frac{7}{2}m + \frac{1}{2}m^2$$

edges. Theorem 1 gives an upper bound of

$$(k+1)(3(4k+4+m) - 4k - 6) = 8k^2 + 14k + 3mk + 6 + 3m$$

edges in a k-visibility graph with $4k + 4 + m$ vertices. Hence

$$8k^2 + 14k + 4mk + 6 + \frac{7}{2}m + \frac{1}{2}m^2 \leq 8k^2 + 14k + 3mk + 6 + 3m$$

$$4mk + \frac{7}{2}m + \frac{1}{2}m^2 \leq 3mk + 3m$$

$$mk + \frac{1}{2}(m + m^2) \leq 0,$$

and therefore $m \leq 0$. Hence $n \leq 4k + 4$. □

Dean et al. [4] gave a construction achieving this edge bound for all $n \geq 4k + 4$ (see Figure 1). Notice when $n = 4k + 4$, the construction is the complete graph K_{4k+4}. When $n < 4k + 4$, the k-visibility graph with the most edges is a complete graph on n vertices, obtained by leaving any $4k + 4 - n$ bars out of the K_{4k+4} representation.

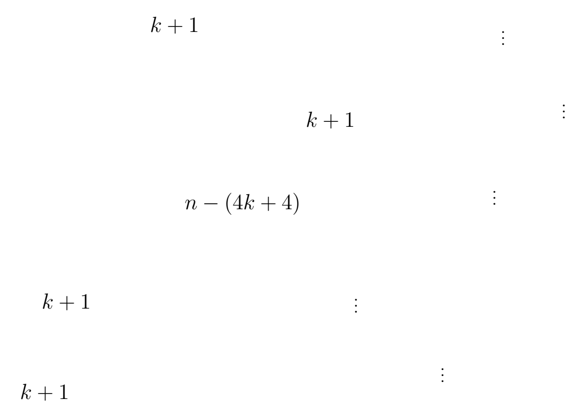

Figure 1: A bar k-visibility representation with $(k+1)(3n-4k-6)$ edges.

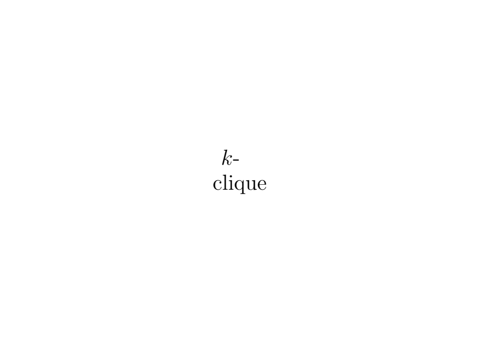

Figure 2: Example: W_6^k

3 Comparing the Families $\mathcal{F}_{k-1}$ and $\mathcal{F}_k$

Corollary 2 shows that $K_{4k+4} \in \mathcal{F}_k$ but not $\mathcal{F}_{k-1}$. A natural question is whether $\mathcal{F}_{k-1}$ is contained in $\mathcal{F}_k$. In order to answer this question, we will need the following lemma:

Lemma 3. *Suppose in some bar k-visibility representation of a graph G, $I(v) \cap I(w) \neq \emptyset$ but $v \not\leftrightarrow w$. Then for any vertical line ℓ intersecting $I(v) \cap I(w)$, if ℓ crosses $B(x)$, x is contained in a $(k+2)$-clique whose intervals also intersect ℓ.*

Proof. If $v \not\leftrightarrow w$, then there must be at least $k+1$ bars blocking $B(v)$ from $B(w)$. Any consecutive $k+2$ bars along ℓ, including $B(x)$, can all see each other, and hence their associated vertices must form a $(k+2)$-clique. □

Define a k-wheel W_n^k to be the graph obtained by joining every vertex of a k-clique with every vertex of an n-cycle (see Figure 2).

Proposition 4. *For $n \geq 5$, W_n^k is not a bar k-visibility graph.*

$\lfloor n/2 \rfloor$ intervals

k-clique

$\lceil n/2 \rceil$ intervals

Figure 3: A bar $(k-1)$-visibility representation of W_n^k.

Proof. Since W_n^k contains an induced long cycle C_n, it is not an interval graph. Therefore there must be two vertices v and w such that $I(v) \cap I(w) \neq \emptyset$ but $v \not\leftrightarrow w$. Let $I(v) \cap I(w) = [a, b]$. As the only $(k+2)$-clique containing v or w contains the middle k-clique, then by Lemma 3 any vertical line intersecting $[a, b]$ must intersect all k bars of the k-clique. The k-clique is not sufficient to obstruct $B(v)$'s view of $B(w)$, hence there must be another bar located between them. Let $B(x)$ be the first bar *not* associated with the middle k-clique that is intersected by a vertical line drawn from v to w. Note that x is one of v's two neighbors in C_n. Let $I(x) = [a', b']$. Let v' be v's other neighbor in C_n; note that $v' \neq w$ and $v' \not\leftrightarrow x$. Now, $n \geq 5$ implies that v' and x have no common neighbor on the cycle, so there can be no $(k+1)$-clique between $B(x)$ and $B(v')$. Therefore we must have $I(v') \cap I(x) = \emptyset$. Since v and v' must be adjacent, $I(v) \cap I(v') \neq \emptyset$; assume by symmetry that $I(v) \cap I(v') = [c, d]$ for some $d < a'$. Since deleting v, x, and the k-clique from W_n^k leaves a connected graph, there must be some interval $I(z)$ that intersects $[d, a']$, where z is not v, x, or a vertex of the k-clique. The bar $B(z)$ closest to $B(v)$ with $I(z)$ intersecting $[d, a']$ must be visible to $B(v)$, since only the bars corresponding to the vertices of the k-clique could be located between $B(v)$ and $B(z)$. Since v has no other neighbors, this is a contradiction. □

Note that an easy construction shows that W_4^k is a bar k-visibility graph, so the result is sharp.

The results above combine to give the following:

Theorem 5. *For all k, the families $\mathcal{F}_k$ and $\mathcal{F}_{k-1}$ are incomparable under inclusion.*

Proof. $\mathcal{F}_k \nsubseteq \mathcal{F}_{k-1}$ follows from Corollary 2. For the reverse inclusion, we observe that $W_n^k \in \mathcal{F}_{k-1}$; Figure 3 gives a bar $(k-1)$-visibility representation. By Proposition 4, $W_n^k \notin \mathcal{F}_k$, hence $\mathcal{F}_{k-1} \nsubseteq \mathcal{F}_k$. □

4 Induced Subgraphs

We have already observed that as k increases, $\mathcal{F}_k$ approaches the family of interval graphs. It is known that all interval graphs are chordal graphs and interval graphs do not contain any induced subdivided complete bipartite graph $K_{1,3}$ [5]. We have shown already that W_n^k is a $(k-1)$-visibility graph and hence k-visibility graphs may contain induced long cycles.

k-clique

k-clique

Figure 4: A bar k-visibility representation of a graph with an induced subdivided $K_{1,3}$.

One may wonder whether an induced subdivision of $K_{1,3}$ prevents a graph from being in $\mathcal{F}_k$ for any k. The following proposition answers this question.

Proposition 6. *For every tree T and every $k \geq 0$, there exists a graph G such that G contains T as an induced subgraph and G is a bar k-visibility graph.*

Proof. Choose a vertex r of T to be the root, and fix some integer d. We define the placement of $V(T)$'s bars in a k-visibility representation inductively. Assign the root the bar $B(r)$. Having assigned bars to all vertices at distance ℓ from the root, we place the bars for the vertices at distance $\ell + 1$ as follows: For a vertex v at level ℓ, find its children $v_1, \ldots, v_m$. Divide $B(v)$ into $2m - 1$ closed segments, and assign v_i the $(2i - 1)^{st}$ segment. Translate this segment down a distance of d to obtain $B(v_i)$.

Having assigned bars to all vertices of T in this way, if v and w are adjacent in T, then $I(v) \cap I(w) \neq \emptyset$. By placing a k-clique between each level, we ensure that only vertices at adjacent levels can see each other. The graph induced by this bar k-visibility representation is the desired graph G. □

Figure 4 gives an example when T is an induced subdivided $K_{1,3}$.

We prove instead that certain nonplanar subgraphs are forbidden as induced subgraphs.

Proposition 7. *Suppose that a graph G contains a triangle-free nonplanar induced subgraph. Then G is not a bar k-visibility graph for any k.*

Proof. Suppose that G is a bar k-visibility graph for some k and has a triangle-free nonplanar induced subgraph H. Fix a bar k-visibility representation of G. Any two adjacent vertices of H are also adjacent in G, and thus their associated intervals intersect in the bar k-visibility representation of G. A pair of adjacent vertices u and v must exist in H such that any vertical line segment joining $B(u)$ and $B(v)$ intersects the bar of at least one other vertex w in H. Otherwise, if we restrict the bar k-visibility representation of G to the vertices in H, we would obtain a planar representation of H. However, by assumption, H is nonplanar. Thus, when $B(u)$ sees $B(v)$ in the bar k-visibility representation of G, the line of sight intersects $B(w)$. Therefore u, v, and w form a triangle in G which will also be in H. This contradicts the assumption that H is a triangle-free induced subgraph of G, and hence G cannot be a bar k-visibility graph for any k. □

5 Regular Bar k-Visibility Graphs

It is easy to show that the only connected regular interval graphs are complete graphs. For small degrees, this fact remains true for bar k-visibility graphs.

Proposition 8. *If G is a connected d-regular bar k-visibility graph with $d \leq 2k+1$, then G is a complete graph.*

Proof. Let v be a vertex whose bar begins last; that is, no vertex has a bar whose left endpoint is farther right than v's. Let $I(v) = [a, b]$, and let $v_1, v_2, \ldots, v_m$ be the vertices whose intervals contain the point a, where the vertices are ordered from top to bottom by the height of their corresponding bars. Note that $v = v_i$ for some i.

All of v's neighbors are among $v_1, \ldots, v_m$, so $\deg(v) \leq m-1$. Since $v_{\lfloor m/2 \rfloor}$ can see $k+1$ bars above it and $k+1$ bars below it if enough bars are present, $\deg(v_{\lfloor m/2 \rfloor}) \geq m-1$. Hence, $d = \deg(v) = \deg(v_{\lfloor m/2 \rfloor}) = m-1$.

If G is not a complete graph, there exists at least one vertex whose bar ends before v's bar begins. Among all such vertices, let c be the maximum value of a right endpoint of the associated intervals. Note that $c < a$. Let $z_1, z_2, \ldots, z_p$ be all the vertices whose intervals' right endpoints are c. As G is connected, some v_i must be adjacent to some z_j. Among the bars $B(v_1), \ldots, B(v_m)$ seeing some $B(z_j)$ at the point c, choose i to minimize $|\lfloor m/2 \rfloor - i|$.

We claim that $\deg(v_i) \geq m$. We know that z_j is adjacent to v_i; suppose some v_ℓ is not in v_i's neighborhood, $i \neq \ell$. But then $c \in I(v_\ell)$, since any bar that begins after c must see $B(v_i)$ in order to have degree $m-1$. Consider the point $c+\epsilon$, where ϵ is chosen to be small enough such that no interval begins in $[c, c+\epsilon]$. As v_i was chosen to be the "most central" bar extending left to the point c, there cannot be k intervals containing the point $c+\epsilon$ blocking $B(v_i)$ from $B(v_\ell)$. Therefore $v_i \leftrightarrow v_j$, and hence $\deg(v_i) \geq m$, contradicting the assumption that G is regular. □

When $d = 2k+2$ and $k \in \{0, 1, 2, 3, 4\}$, there exist d-regular non-cliques in $\mathcal{F}_k$. For $k = 0$, every cycle C_n with $n \geq 4$ is a 2-regular bar 0-visibility graph. Figure 5 shows non-cliques that are $2k+2$ regular when $k = 1$ and $k = 2$. Figures 6 and 7 show constructions for an infinite number of regular graphs of degree $2k+2$ when $k = 3$ and $k = 4$, respectively.

The question remains open for larger values of d and k.

6 Conclusion

There are many open questions remaining about bar k-visibility graphs, with the primary goal being a complete characterization of bar k-visibility graphs. There are also several other interesting questions that may serve as intermediate steps toward this goal.

1. Are there forbidden induced subgraphs for bar k-visibility graphs besides triangle-free nonplanar graphs?

Figure 5: On the left is a bar 1-visibility representation of the 4-regular graph formed by removing a perfect matching from K_6. On the right is a bar 2-visibility representation of the 6-regular graph formed by removing a perfect matching from K_8.

Figure 6: A bar 3-visibility representation of an 8-regular graph. The 9 bars in the repeatable block can be repeated horizontally as many times as desired, including omitting it entirely. Consecutive blocks may need to be perturbed vertically a small amount so that the top and bottom bars can see the top and bottom bars from the next block, but are still disjoint from those bars.

repeatable block

Figure 7: A bar 4-visibility representation of a 10-regular graph. The 11 bars in the repeatable block can be repeated horizontally as many times as desired, including omitting it entirely.

2. Does every graph that is not a bar k-visibility graph for any k contain an induced triangle-free nonplanar subgraph?

3. Are there $(2k+2)$-regular bar k-visibility graphs for $k \geq 5$?

4. Are there d-regular bar k-visibility graphs with $d \geq 2k+3$?

Dean et al. [4] also present several open questions regarding the chromatic number, genus, and thickness of bar k-visibility graphs. It is worth noting that while we now have a sharp edge bound, it does not improve the upper bound of $6k+6$ in [4] for the chromatic number of bar k-visibility graphs. We feel that this bound can be lowered, possibly through a deeper exploration of the structural aspects of bar k-visibility graphs and their connection to minimum degree and degeneracy.

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