Exam 1 Solutions

Math 231

February 22, 2007

1. (a) Write the general formula for integration by parts. (3 pts)

   SOLUTION: $\int udv = uv - \int vdu$

   (b) Compute $\int \sin(2x)e^{2x}\, dx$. (8 pts)

   SOLUTION: Use integration by parts twice. $u = \sin 2x$, $dv = e^{2x}\, dx$ so $du = 2\cos 2x$ and $v = \frac{1}{2}e^{2x}$. This gives

$$\int \sin(2x)e^{2x}\, dx = \frac{1}{2}\sin(2x)e^{2x} - \int \cos(2x)e^{2x}\, dx.$$

   We now do integration by parts again with $u = \cos 2x$, $dv = e^{2x}\, dx$ so $du = -2\sin 2x$. This gives

$$\int \sin(2x)e^{2x}\, dx = \frac{1}{2}\sin(2x)e^{2x} - \left[\frac{1}{2}\cos(2x)e^{2x} - \int -\sin(2x)e^{2x}\, dx\right].$$

   Solving for the integral in question we get

$$\int \sin(2x)e^{2x}\, dx = \frac{1}{2}\left[\frac{1}{2}\sin(2x)e^{2x} - \frac{1}{2}\cos(2x)e^{2x}\right] + C.$$

2. (a) Draw the reference triangle for the substitution $4x = 5\sin\theta$. (3 pts)

   SOLUTION: This is a right triangle with sides $4x$, 5, and hypotenuse $\sqrt{16x^2+25}$.

   (b) Evaluate $\int \frac{1}{x\sqrt{(\ln x)^2 - 9}}\, dx$. (8 pts)

   SOLUTION: We use the trigonometric solution $\ln x = 3\sec\theta$, $\frac{1}{x}\, dx = 3\sec\theta\tan\theta\, d\theta$. This makes the new integral

$$\int \frac{1}{x\sqrt{(\ln x)^2 - 9}}\, dx = \int \frac{3\sec\theta\tan\theta}{\sqrt{9\sec^2\theta - 9}}\, d\theta = \int \frac{3\sec\theta\tan\theta}{\sqrt{9\tan^2\theta - 9}}\, d\theta = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\, d\theta.$$

   Using the fact on the first page we complete the integral:

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Using the reference triangle we finish the problem:

$$\int \frac{1}{x\sqrt{(\ln x)^2 - 9}}\, dx = \ln\left|\frac{\ln x}{3} + \frac{\sqrt{(\ln x)^2 - 9}}{3}\right| + C.$$

3. (a) Evaluate $\int \sin^4(x)\, dx$. (8 pts)

SOLUTION: We use the half angle formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

$$\int \sin^4(x)\, dx = \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 dx = \int \left[\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right] dx$$

Now use the half angle formula $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$ to get

$$\begin{aligned}\int \sin^4(x)\, dx &= \int \left[\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}(1 + \cos 4x)\right] dx \\ &= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C\end{aligned}$$

(b) Evaluate $\int \tan^3 x \sec x\, dx$. (8 pts)

SOLUTION: We use the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$.

$$\begin{aligned}\int \tan^3 x \sec x\, dx &= \int \tan^2 x \tan x \sec x\, dx \\ &= \int (\sec^2 x - 1) \sec x \tan x\, dx \\ &= \int (\sec^3 x - \sec x) \sec x \tan x\, dx\end{aligned}$$

We use the u substitution $u = \sec x$, $du = \sec x \tan x\, dx$ and the integral becomes:

$$\int (u^3 - u)\, du = \frac{1}{2}u^4 - \frac{1}{2}u^2 + C = \sec^4 x - \sec^2 x + C.$$

4. (a) Find the general form of the partial fraction decomposition of $\dfrac{x^2 - 2}{(x^2 + 4x + 4)(x^2 + 1)}$ (do not find the actual values of A, B, etc.). (3pts)

SOLUTION:

$$\frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{Cx + D}{x^2 + 1}$$

(b) Find the partial fraction decomposition of $\frac{4x^2+3x+2}{x^3+2x^2+x}$. (6 pts)
SOLUTION: The partial fraction decomposition has the form

$$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}.$$

This leads to the system of equations

$$A+B=4$$
$$2A+B+C=3$$
$$A=2$$

which yields the solution $A=2, B=2, C=-3$, so the partial fraction decomposition is

$$\frac{2}{x}+\frac{2}{x+1}-\frac{3}{(x+1)^2}.$$

(c) Evaluate $\int \frac{4x^2+3x+2}{x^3+2x^2+x}\,dx$. (4 pts)
SOLUTION:

$$\begin{aligned}\int \frac{4x^2+3x+2}{x^3+2x^2+x}\,dx &= \int \frac{2}{x}+\frac{2}{x+1}-\frac{3}{(x+1)^2}\,dx \\ &= 2\ln|x|+2\ln|x+1|+\frac{3}{x+1}+C\end{aligned}$$

5. Evaluate $\int \frac{1}{\sqrt{-x^2-6x-5}}\,dx$. Please work out any trigonometric substitutions that may be necessary. (10 pts)

SOLUTION: First we complete the square to get $-x^2-6x-5=4-(x+3)^2$ Now use the trig substitution $x+3=2\sin\theta\; dx=2\cos\theta\,d\theta$.

$$\int \frac{1}{\sqrt{-x^2-6x-5}}\,dx=\int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta=\int 1\,d\theta=\theta+C$$

We then use the reference triangle to finish the integral.

$$\int \frac{1}{\sqrt{-x^2-6x-5}}\,dx=\arcsin\left(\frac{x+3}{2}\right)+C.$$

6. Determine if $\int_{-8}^{8} x^{-2/3}\,dx$ converges or diverges and evaluate the integral if it converges. (10 pts)

SOLUTION: This problem has an infinite integrand at $x=0$, so we must separate the integral:

$$\int_{-8}^{8} x^{-2/3}\,dx=\int_{-8}^{0} x^{-2/3}\,dx+\int_{0}^{8} x^{-2/3}\,dx$$

We then must use directional limits and we get

$$\begin{aligned}\int_{-8}^{8} x^{-2/3}\,dx &= \int_{-8}^{0} x^{-2/3}\,dx + \int_{0}^{8} x^{-2/3}\,dx \\ &= \lim_{a\to 0^-} 3x^{1/3}\Big|_{-8}^{a} + \lim_{b\to 0^+} 3x^{1/3}\Big|_{b}^{8} \\ &= \lim_{a\to 0^-} 3a^{1/3} - 3(-8)^{1/3} + \lim_{b\to 0^+} 3(8)^{1/3} - 3(b)^{1/3} \\ &= -3(-8)^{1/3} + 3(8)^{1/3} = -3(-2) + 3(2) = 12.\end{aligned}$$

Thus the integral converges.

7. Determine the convergence of $\int_1^\infty x^{-2}\ln x\,dx$ and evaluate the integral if it converges. (9 pts)

SOLUTION: This improper integral requires integration by parts as the first step. We select $u = \ln x, dv = x^{-2}\,dx$, and we get $du = x^{-1}\,dx, v = -x^{-1}$. We now have

$$\begin{aligned}\int_1^\infty x^{-2}\ln x\,dx &= \frac{\ln x}{x}\Big|_1^\infty - \int_1^\infty -x^{-2}\,dx \\ &= \lim_{a\to\infty} \frac{\ln x}{x}\Big|_1^a - \frac{1}{x}\Big|_1^a \\ &= \lim_{a\to\infty} \frac{\ln a}{a} - \frac{\ln 1}{1} - \left[\frac{1}{a} - \frac{1}{1}\right] \\ &= 1\end{aligned}$$

Thus the integral converges. The limit used the fact that was given on the front of the page that $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.