Exam 3

Math 231

April 23, 2007

Name: [redacted] KEY

Directions

1. Be sure to show all work to receive full credit.
2. No calculators, books, notes, ipods or cell phones are allowed during the exam.
3. To receive full credit, you must state all rules that you use.

1. (a) (6 pts) Determine if the sequence $a_n = \dfrac{\sin n}{n}$ converges or diverges. If it converges, compute its limit. Be sure to show all work.

Squeeze law:

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{-1}{n} = 0$$

So $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ and the sequence converges.

(b) (6 pts) Use the bounded monotonic sequence property to show that the sequence $a_n = 1 + \dfrac{1}{n^2}$ converges.

Bounded: $2 \geq 1+\frac{1}{n^2} \geq 1 \quad \forall n.$

Monotonic: $1+\frac{1}{n^2} - \left(1+\frac{1}{(n+1)^2}\right) = \frac{1}{n^2} - \frac{1}{(n+1)^2} > 0$

So $1+\frac{1}{n^2} \geq 1+\frac{1}{(n+1)^2}$.

The sequence is bounded and monotonic decreasing so it converges.

2. (a) (4 pts) Find the third and fourth partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$S_3 = \sum_{n=1}^{3} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \sum_{n=1}^{4} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

(b) (3 pts) Write down a formula for the kth partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$S_k = \sum_{n=1}^{k} \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

(c) (5 pts) Use partial sums to show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

We check the limit of the partial sums.

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{2^k - 1}{2^k} = \lim_{k\to\infty} 1 - \frac{1}{2^k} = 1.$$

Thus $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1.

3. (a) (3 pts) Define what it means for an alternating series to converge conditionally.

$\sum (-1)^n a_n$ converges conditionally if

- $\sum (-1)^n a_n$ converges
- $\sum |(-1)^n a_n|$ diverges

(b) (8 pts) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+\ln n)}{n^4+n-2}$ diverges, converges absolutely or converges conditionally.

Check for absolute convergence:

$$\sum_{n=1}^{\infty} \frac{n^2+\ln n}{n^4+n-2}$$

Limit Comparison test.

$\sum \frac{1}{n^2}$ converges by the p-test

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{n^2+\ln n}{n^4+n-2}} = \lim_{n\to\infty} \frac{n^4+n-2}{n^4+n^2\ln n} = \lim_{n\to\infty} \frac{n^4/n^4 + n/n^4 - 2/n^4}{n^4/n^4 + \frac{n^2 \ln n}{n^4}}$$

$$= \lim_{n\to\infty} \frac{1+1/n^3 - 2/n^4}{1+\frac{\ln n}{n^2}} = 1.$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)(n^2+\ln n)}{n^4+n-2}$ converges absolutely

by ~~the compar~~ the limit comparison test.

5. (a) (3 pts) Write down the Maclaurin series for the function $f(x) = \sin x$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(b) (5 pts) Find the Maclaurin series for the function $g(x) = \dfrac{\sin t - t}{t^2}$.

$$\frac{\sin t - t}{t^2} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2t+1)!} - t}{t^2} = \frac{\sum_{n=1}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}}{t^2} = \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n-1}}{(2n+1)!}$$

OR: $$\frac{\left(\cancel{t} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right) - t}{t^2} = \frac{-t^3/3! + t^5/5! - t^7/7! + \ldots}{t^2} = -t/3! + t^3/5! - t^5/7! \ldots$$

(c) (6 pts) Find a power series representation of the function $f(x) = \displaystyle\int_0^x \frac{\sin t - t}{t^2}\, dt$.

$$\int_0^x \frac{\sin t - t}{t^2}\, dt = \int_0^x -t/3! + t^3/5! - t^5/7! + \ldots$$

$$= \left. \frac{-t^2}{2 \cdot 3!} + \frac{t^4}{4 \cdot 5!} - \frac{t^6}{6 \cdot 7!} + \ldots \right]_0^x$$

OR:

$$\int_0^x \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n-1}}{(2n+1)!} = \left. \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2n(2n+1)!} \right]_0^x$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n+1)!}$$

OR $$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)(2n+3)!}$$

6. (11 pts) Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{\ln(n+1)}$.

Ratio test: $\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x+1)^{n+1}}{\ln(n+2)}}{\frac{(-1)^n(x+1)^n}{\ln(n+1)}} \right| = \lim_{n\to\infty} \left| \frac{(x+1)\ln(n+1)}{\ln(n+2)} \right|$

$= \lim_{n\to\infty} |x+1| \frac{\ln(n+1)}{\ln(n+2)} \overset{L'H}{=} |x+1| \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}}$

$= |x+1| \lim_{n\to\infty} \frac{n/n + 2/n}{n/n + 1/n} = |x+1|$

$|x+1| < 1 \implies$ Radius $= 1$, $-2 < x < 0$

Check endpoints:

$x = -2$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-2+1)^n}{\ln(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\ln(n+1)} = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

Diverges by comparison test with $\sum \frac{1}{n}$

$x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{\ln(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ Converges by the Alternating Series Test.

$R = 1$, $I = (-2, 0]$.

7. (8 pts) Find the second degree Taylor polynomial of $f(x) = \ln \ln x$ at $x = 2$. There is no need to simplify the coefficients of the polynomial.

$f(x) = \ln \ln x$

$f'(x) = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$

$f''(x) = -(x \ln x)^{-2}(x \frac{1}{x} + \ln x) = \frac{-(1+\ln x)}{(x \ln x)^2}$

$f(2) = \ln \ln 2$

$f'(2) = \frac{1}{2 \ln 2}$

$f''(2) = \frac{-(1+\ln 2)}{(2 \ln 2)^2}$

$P_3(x) = \ln \ln 2 + \frac{(x-2)}{2 \ln 2} - \frac{(1+\ln 2)}{(2 \ln 2)^2} \frac{(x-2)^2}{2!}$