Chapter 2

The Elementary Theory

2.1 Integration on Paths

The integral of a complex-valued function on a path in the complex plane will be introduced via the integral of a complex-valued function of a real variable, which in turn is expressed in terms of an ordinary Riemann integral.

2.1.1 Definition

Let $\varphi : [a, b] \to \mathbb{C}$ be a piecewise continuous function on the closed interval $[a, b]$ of reals. The *Riemann integral* of φ is defined in terms of the real and imaginary parts of φ by

$$\int_a^b \varphi(t)\, dt = \int_a^b \operatorname{Re} \varphi(t)\, dt + i \int_a^b \operatorname{Im} \varphi(t)\, dt.$$

2.1.2 Basic Properties of the Integral

The following linearity property is immediate from the above definition and the corresponding result for real-valued functions:

$$\int_a^b (\lambda\varphi(t) + \mu\psi(t))\, dt = \lambda \int_a^b \varphi(t)\, dt + \mu \int_a^b \psi(t)\, dt$$

for any complex numbers λ and μ. A slightly more subtle property is

$$\left| \int_a^b \varphi(t)\, dt \right| \le \int_a^b |\varphi(t)|\, dt.$$

This may be proved by approximating the integral on the left by Riemann sums and using the triangle inequality. A somewhat more elegant argument uses a technique called polarization, which occurs quite frequently in analysis. Define $\lambda = \left| \int_a^b \varphi(t) d\, t \right| / \int_a^b \varphi(t)\, dt$; then $|\lambda| = 1$. (If the denominator is zero, take λ to be any complex number of absolute value 1.) Then $|\int_a^b \varphi(t)\,dt| = \lambda\int_a^b \varphi(t)\,dt = \int_a^b \lambda\varphi(t)dt$ by linearity. Since the absolute value of a complex number is real,

$$\left|\int_a^b \varphi(t)\,dt\right| = \operatorname{Re}\int_a^b \lambda\varphi(t)\,dt = \int_a^b \operatorname{Re}\lambda\varphi(t)\,dt$$

by definition of the integral. But $\operatorname{Re}|z| \le |z|$, so

$$\int_a^b \operatorname{Re}\lambda\varphi(t)\,dt \le \int_a^b |\lambda\varphi(t)|\,dt = \int_a^b |\varphi(t)|\,dt$$

because $|\lambda| = 1$. ♣

The fundamental theorem of calculus carries over to complex-valued functions. Explicitly, if φ has a continuous derivative on $[a,b]$, then

$$\varphi(x) = \varphi(a) + \int_a^x \varphi'(t)\,dt$$

for $a \le x \le b$. If φ is continuous on $[a,b]$ and $F(x) = \int_a^x \varphi(t)\,dt, a \le x \le b$, then $F'(x) = \varphi(x)$ for all x in $[a,b]$. These assertions are proved directly from the corresponding results for real-valued functions.

2.1.3 Definition

A *curve* in $\mathbb{C}$ is a continuous mapping γ of a closed interval $[a,b]$ into $\mathbb{C}$. If in addition, γ is piecewise continuously differentiable, then γ is called a *path*. A curve (or path) with $\gamma(a) = \gamma(b)$ is called a *closed curve* (or path). The range (or image or trace) of γ will be denoted by γ^*. If γ^* is contained in a set S, γ is said to be a *curve* (or *path*) *in* S.

Intuitively, if $z = \gamma(t)$ and t changes by a small amount dt, then z changes by $dz = \gamma'(t)\,dt$. This motivates the definition of the *length* L of a path γ:

$$L = \int_a^b |\gamma'(t)|\,dt$$

and also motivates the following definition of the *path integral* $\int_\gamma f(z)dz$.

2.1.4 Definition

Let $\gamma : [a,b] \to \mathbb{C}$ be a path, and let f be continuous on γ, that is, $f : \gamma^* \to \mathbb{C}$ is continuous. We define the integral of f on (or along) γ by

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

It is convenient to define $\int_\gamma f(z)\,dz$ with γ replaced by certain point sets in the plane. Specifically, if $[z_1, z_2]$ is a line segment in $\mathbb{C}$, we define

$$\int_{[z_1,z_2]} f(z)\,dz = \int_\gamma f(z)\,dz$$

where $\gamma(t) = (1-t)z_1 + tz_2, 0 \le t \le 1$. More generally, if $[z_1, \dots, z_{n+1}]$ is a polygon joining z_1 to z_{n+1}, we define

$$\int_{[z_1,z_2,\dots,z_{n+1}]} f(z)\,dz = \sum_{j=1}^{n} \int_{[z_j,z_{j+1}]} f(z)\,dz.$$

The next estimate will be referred to as the *M-L theorem.*

2.1.5 Theorem

Suppose that f is continuous on the path γ and $|f(z)| \le M$ for all $z \in \gamma^*$. If L is the length of the path γ, then

$$\left| \int_\gamma f(z)\,dz \right| \le ML.$$

Proof. Recall from (2.1.2) that the absolute value of an integral is less than or equal to the integral of the absolute value. Then apply the definition of the path integral in (2.1.4) and the definition of length in (2.1.3). ♣

The familiar process of evaluating integrals by anti-differentiation extends to integration on paths.

2.1.6 Fundamental Theorem for Integrals on Paths

Suppose $f : \Omega \to \mathbb{C}$ is continuous and f has a *primitive* F on Ω, that is, $F' = f$ on Ω. Then for any path $\gamma : [a, b] \to \Omega$ we have

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is a closed path in Ω, then $\int_\gamma f(z)\,dz = 0$.

Proof. $\int_\gamma f(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt}F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a))$ by the fundamental theorem of calculus [see (2.1.2)]. ♣

2.1.7 Applications

(a) Let $z_1, z_2 \in \mathbb{C}$ and let γ be any *path from* z_1 *to* z_2, that is, $\gamma : [a, b] \to \mathbb{C}$ is any path such that $\gamma(a) = z_1$ and $\gamma(b) = z_2$. Then for $n = 0, 1, 2, 3, \dots$ we have

$$\int_\gamma z^n\,dz = (z_2^{n+1} - z_1^{n+1})/(n+1).$$

This follows from (2.1.6) and the fact that $z^{n+1}/(n+1)$ is a primitive of z^n. The preceding remains true for $n = -2, -3, -4, \dots$ provided that $0 \notin \gamma^*$ and the proof is the same: $z^{n+1}/(n+1)$ is a primitive for z^n on $\mathbb{C} \setminus \{0\}$. But if $n = -1$, then the conclusion may fail as the following important computation shows. Take $\gamma(t) = e^{it}, 0 \le t \le 2\pi$ (the unit circle, traversed once in the positive sense). Then

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i \ne 0.$$

This also shows that $f(z) = 1/z$, although analytic on $\mathbb{C} \setminus \{0\}$, does not have a primitive on $\mathbb{C} \setminus \{0\}$.

(b) Suppose f is analytic on the open connected set Ω and $f'(z) = 0$ for all $z \in \Omega$. Then f is constant on Ω.

Proof. Let $z_1, z_2 \in \Omega$. Since Ω is polygonally connected, there is a (polygonal) path $\gamma : [a,b] \to \Omega$ such that $\gamma(a) = z_1$ and $\gamma(b) = z_2$. by (2.1.6), $\int_\gamma f'(z)\,dz = f(z_2) - f(z_1)$. But the left side is zero by hypothesis, and the result follows. ♣

Remark

If we do not assume that Ω is connected, we can prove only that f restricted to any component of Ω is constant.

Suppose that a continuous function f on Ω is given. Theorem 2.1.6 and the applications following it suggest that we should attempt to find conditions on f and/or Ω that are sufficient to guarantee that f has a primitive. Let us attempt to imitate the procedure used in calculus when f is a real-valued continuous function on an open interval in $\mathbb{R}$. We begin by assuming Ω is starlike with star center z_0, say. Define F on Ω by

$$F(z) = \int_{[z_0,z]} f(w)\,dw.$$

If $z_1 \in \Omega$, let us try to show that $F'(z_1) = f(z_1)$. If z is near but unequal to z_1, we have

$$\frac{F(z) - F(z_1)}{z - z_1} = \frac{1}{z - z_1}\left(\int_{[z_0,z]} f(w)\,dw - \int_{[z_0,z_1]} f(w)\,dw\right)$$

and we would like to say, as in the real variables case, that

$$\int_{[z_0,z]} f(w)\,dw - \int_{[z_0,z_1]} f(w)\,dw = \int_{[z_1,z]} f(w)\,dw, \tag{1}$$

from which it would follow quickly that $(F(z) - F(z_1))/(z - z_1) \to f(z_1)$ as $z \to z_1$. Now if T is the triangle $[z_0, z_1, z, z_0]$, equation (1) is equivalent to the statement that $\int_T f(w)\,dw = 0$, but as the example at the end of (2.1.7(a)) suggests, this need not be true, even for analytic functions f. However, in the present setting, we can make the key observation that if $\hat{T}$ is the union of T and its interior (the *convex hull* of T), then $\hat{T} \subseteq \Omega$. If f is analytic on Ω, it must be analytic on $\hat{T}$, and in this case, it turns out that $\int_T f(w)\,dw$ *does* equal 0. This is the content of Theorem 2.1.8; a somewhat different version of this result was first proved by Augustin-Louis Cauchy in 1825.

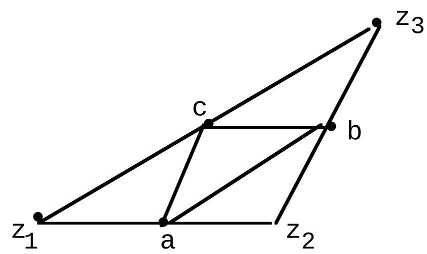

Figure 2.1.1

2.1.8 Cauchy's Theorem for Triangles

Suppose that f is analytic on Ω and $T = [z_1, z_2, z_3, z_1]$ is any triangle such that $\hat{T} \subseteq \Omega$. Then $\int_T f(z)\,dz = 0$.

Proof. Let a, b, c be the midpoints of $[z_1, z_2], [z_2, z_3]$ and $[z_3, z_1]$ respectively. Consider the triangles $[z_1, a, c, z_1], [z_2, b, a, z_2], [z_3, c, b, z_3]$ and $[a, b, c, a]$ (see Figure 2.1.1). Now the integral of f on T is the sum of the integrals on the four triangles, and it follows from the triangle inequality that if T_1 is one of these four triangles chosen so that $|\int_{T_1} f(z)\,dz|$ is as large as possible, then

$$|\int_T f(z)\,dz| \le 4|\int_{T_1} f(z)\,dz|.$$

Also, if L measures length, then $L(T_1) = \frac{1}{2}L(T)$, because a line joining two midpoints of a triangle is half as long as the opposite side. Proceeding inductively, we obtain a sequence $\{T_n : n = 1, 2, \dots\}$ of triangles such that $L(T_n) = 2^{-n}L(T),\ \hat{T}_{n+1} \subseteq \hat{T}_n$, and

$$|\int_T f(z)\,dz| \le 4^n|\int_{T_n} f(z)\,dz|. \tag{1}$$

Now the $\hat{T}_n$ form a decreasing sequence of nonempty closed and bounded (hence compact) sets in $\mathbb{C}$ whose diameters approach 0 as $n \to \infty$. Thus there is a point $z_0 \in \cap_{n=1}^{\infty}\hat{T}_n$. (If the intersection is empty, then by compactness, some finite collection of $\hat{T}_i$'s would have empty intersection.) Since f is analytic at z_0, there is a continuous function $\epsilon : \Omega \to \mathbb{C}$ with $\epsilon(z_0) = 0$ [see (5) of (1.3.1)] and such that

$$f(z) = f(z_0) + (z - z_0)[f'(z_0) + \epsilon(z)],\ z \in \Omega. \tag{2}$$

By (2) and (2.1.7a), we have

$$\int_{T_n} f(z)\,dz = \int_{T_n} (z - z_0)\epsilon(z)\,dz,\ n = 1, 2, 3, \dots\,. \tag{3}$$

But by the M-L theorem (2.1.5),

$$
\begin{aligned}
|\int_{T_n}(z-z_0)\epsilon(z)\,dz| &\leq \sup_{z\in T_n}[|\epsilon(z)|\;|z-z_0|]L(T_n)\\
&\leq \sup_{z\in T_n}|\epsilon(z)|(L(T_n))^2 \text{ since } z\in \hat{T}_n\\
&\leq \sup_{z\in T_n}|\epsilon(z)|4^{-n}(L(T))^2\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Thus by (1) and (3),

$$
|\int_T f(z)\,dz| \leq \sup_{z\in T_n}|\epsilon(z)|(L(T))^2 \to 0
$$

as $n\to\infty$, because $\epsilon(z_0)=0$. We conclude that $\int_T f(z)\,dz = 0$. ♣

We may now state formally the result developed in the discussion preceding Cauchy's theorem.

2.1.9 Cauchy's Theorem for Starlike Regions

Let f be analytic on the starlike region Ω. Then f has a primitive on Ω, and consequently, by (2.1.6), $\int_\gamma f(z)\,dz = 0$ for every closed path γ in Ω.

Proof. Let z_0 be a star center for Ω, and define F on Ω by $F(z) = \int_{[z_0,z]} f(w)\,dw$. It follows from (2.1.8) and discussion preceding it that F is a primitive for f. ♣

We may also prove the following converse to Theorem (2.1.6).

2.1.10 Theorem

If $f:\Omega\to\mathbb{C}$ is continuous and $\int_\gamma f(z)\,dz = 0$ for every closed path γ in Ω, then f has a primitive on Ω.

Proof. We may assume that Ω is connected (if not we can construct a primitive of f on each component of Ω, and take the union of these to obtain a primitive of f on Ω). So fix $z_0\in\Omega$, and for each $z\in\Omega$, let γ_z be a polygonal path in Ω from z_0 to z. Now define F on Ω by $F(z) = \int_{\gamma_z} f(w)\,dw, z\in\Omega$. Then the discussion preceding (2.1.8) may be repeated without essential change to show that $F' = f$ on Ω. (In Equation (1) in that discussion, $[z_0,z]$ and $[z_0,z_1]$ are replaced by the polygonal paths γ_z and γ_{z_1}, but the line segment $[z_1,z]$ can be retained for all z sufficiently close to z_1.) ♣

2.1.11 Remarks

(a) If $\gamma:[a,b]\to\mathbb{C}$ is a path, we may traverse γ backwards by considering the path λ defined by $\lambda(t) = \gamma(a+b-t),\ a\leq t\leq b$. Then $\lambda^* = \gamma^*$ and for every continuous function

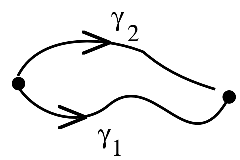

Figure 2.1.2

$f : \gamma^* \to \mathbb{C}$, it follows from the definition of the integral and a brief change of variable argument that

$$\int_\lambda f(z)\,dz = -\int_\gamma f(z)\,dz.$$

(b) Similarly, if $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are paths with $\gamma_1(b) = \gamma_2(c)$, we may attach γ_2 to γ_1 via the path

$$\gamma(t) = \begin{cases} \gamma_1((1-2t)a + 2tb), & 0 \le t \le 1/2 \\ \gamma_2((2-2t)c + (2t-1)d), & 1/2 \le t \le 1. \end{cases}$$

Then $\gamma^* = \gamma_1^* \cup \gamma_2^*$ and for every continuous function $f : \gamma^* \to \mathbb{C}$,

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

There is a technical point that should be mentioned. The path $\gamma_1(t), a \le t \le b$, is strictly speaking not the same as the path $\gamma_1((1-2t)a + 2tb), 0 \le t \le 1/2$, since they have different domains of definition. Given the path $\gamma_1 : [a,b] \to \mathbb{C}$, we are forming a new path $\delta = \gamma_1 \circ h$, where $h(t) = (1-2t)a + 2tb, 0 \le t \le 1/2$. It is true then that $\delta^* = \gamma_1^*$ and for every continuous function f on γ_1^*, $\int_{\gamma_1} f(z)\,dz = \int_\delta f(z)\,dz$. Problem 4 is a general result of this type.

(c) If γ_1 and γ_2 are paths with the same initial point and the same terminal point, we may form a closed path γ by first traversing γ_1 and then traversing γ_2 backwards. If f is continuous on γ^*, then $\int_\gamma f(z)\,dz = 0$ iff $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ (see Figure 2.1.2).

An Application of 2.1.9

Let $\Gamma = [z_1, z_2, z_3, z_4, z_1]$ be a rectangle with center at 0 (see Figure 2.1.3); let us calculate $\int_\Gamma \frac{1}{z}\,dz$. Let γ be a circle that circumscribes the rectangle Γ, and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the arcs of γ joining z_1 to z_2, z_2 to z_3, z_3 to z_4 and z_4 to z_1 respectively. There is an open half plane (a starlike region) excluding 0 but containing both $[z_1, z_2]$ and γ_1^*. By (2.1.9) and Remark (2.1.11c), the integral of $1/z$ on $[z_1, z_2]$ equals the integral of $1/z$ on γ_1. By considering the other segments of Γ and the corresponding arcs of γ, we obtain

$$\int_\Gamma \frac{1}{z}\,dz = \int_\gamma \frac{1}{z}\,dz = \pm 2\pi i$$

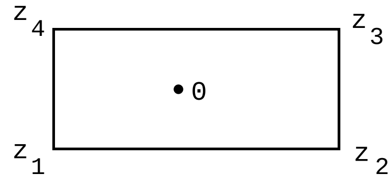

Figure 2.1.3

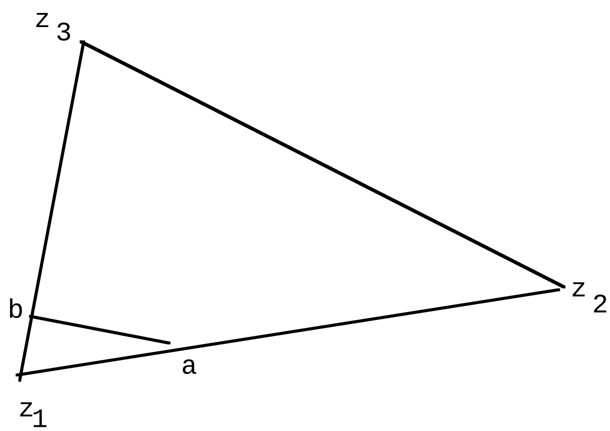

Figure 2.1.4

by a direct calculation, as in (2.1.7a).

The reader who feels that the machinery used to obtain such a simple result is excessive is urged to attempt to compute $\int_\Gamma \frac{1}{z}\,dz$ directly.

The following strengthened form of Cauchy's Theorem for triangles and for starlike regions will be useful in the next section.

2.1.12 Extended Cauchy Theorem for Triangles

Let f be continuous on Ω and analytic on $\Omega \setminus \{z_0\}$. If T is any triangle such that $\hat{T} \subseteq \Omega$, then $\int_T f(z)\,dz = 0$.

Proof. Let $T = [z_1, z_2, z_3, z_1]$. If $z_0 \notin \hat{T}$, the result follows from (2.1.8), Cauchy's theorem for triangles. Also, if z_1, z_2 and z_3 are collinear, then $\int_T f(z)\,dz = 0$ for any continuous (not necessarily analytic) function. Thus assume that z_1, z_2 and z_3 are non-collinear and that $z_0 \in \hat{T}$. Suppose first that z_0 is a vertex, say $z_0 = z_1$. Choose points $a \in [z_1, z_2]$ and $b \in [z_1, z_3]$; see Figure 2.1.4. By (2.1.9),

$$\int_T f(z)\,dz = \int_{[z_1,a]} f(z)\,dz + \int_{[a,b]} f(z)\,dz + \int_{[b,z_1]} f(z).\,dz$$

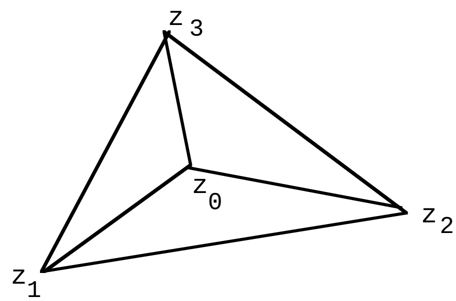

Figure 2.1.5

Since f is continuous at $z_0 = z_1$, each of the integrals on the right approaches zero as $a, b \to z_1$, by the M-L theorem. Therefore $\int_T f(z)\,dz = 0$.

If $z_0 \in \hat{T}$ is not a vertex, join z_0 to each vertex of T by straight line segments (see Figure 2.1.5), and write $\int_T f(z)\,dz$ as a sum of integrals, each of which is zero by the above argument. ♣

2.1.13 Extended Cauchy Theorem for Starlike Regions

Let f be continuous on the starlike region Ω and analytic on $\Omega \setminus \{z_0\}$. Then f has a primitive on Ω, and consequently $\int_\gamma f(z)\,dz = 0$ for every closed path γ in Ω.

Proof. Exactly as in (2.1.9), using (2.1.12) instead of (2.1.8). ♣

Problems

1. Evaluate $\int_{[-i,1+2i]} \operatorname{Im} z\,dz$.
2. Evaluate $\int_\gamma \overline{z}\,dz$ where γ traces the arc of the parabola $y = x^2$ from (1,1) to (2,4).
3. Evaluate $\int_{[z_1,z_2,z_3]} f(z)\,dz$ where $z_1 = -i, z_2 = 2+5i, z_3 = 5i$ and $f(x+iy) = x^2 + iy$.
4. Show that $\int_\gamma f(z)\,dz$ is independent of the parametrization of γ^* in the following sense. Let $h : [c,d] \to [a,b]$ be one-to-one and continuously differentiable, with $h(c) = a$ and $h(d) = b$ (γ is assumed to be defined on $[a,b]$). Let $\gamma_1 = \gamma \circ h$. Show that γ_1 is a path, and prove that if f is continuous on γ^*, then $\int_{\gamma_1} f(z)\,dz = \int_\gamma f(z)\,dz$.
5. In the next section it will be shown that if f is analytic on Ω, then f' is also analytic, in particular continuous, on Ω. Anticipating this result, we can use (2.1.6), the fundamental theorem for integration along paths, to show that $\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a))$. Prove the following.
   (a) If Ω is convex and $\operatorname{Re} f' > 0$ on Ω, then f is one-to-one. (Hint: $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$ implies that $\operatorname{Re}[(f(z_2) - f(z_1))/(z_2 - z_1)] > 0$.)
   (b) Show that (a) does *not* generalize to starlike regions. (Consider $z + 1/z$ on a suitable region.)

(c) Suppose $z_0 \in \Omega$ and $f'(z_0) \neq 0$. Show that there exists $r > 0$ such that f is one-to-one on $D(z_0, r)$. Consequently, if f' has no zeros in Ω, then f is locally one-to-one.

2.2 Power Series

In this section we develop the basic facts about complex series, especially complex power series. The main result is that f is analytic at z_0 iff f can be represented as a convergent power series throughout some neighborhood of z_0. We first recall some elementary facts about complex series in general.

2.2.1 Definition

Given a sequence $w_0, w_1, w_2, \ldots$ of complex numbers, consider the series $\sum_{n=0}^{\infty} w_n$. If $\lim_{n\to\infty} \sum_{k=0}^{n} w_k$ exists and is the complex number w, we say that *the series converges to* w and write $w = \sum_{n=0}^{\infty} w_n$. Otherwise, the series is said to *diverge.*

A useful observation is that a series is convergent iff the partial sums $\sum_{k=0}^{n} w_k$ form a *Cauchy sequence*, that is, $\sum_{k=m}^{n} w_k \to 0$ as $m, n \to \infty$.

The series $\sum_{n=0}^{\infty} w_n$ is said to *converge absolutely* if the series $\sum_{n=0}^{\infty} |w_n|$ is convergent. As in the real variables case, an absolutely convergent series is convergent. A necessary and sufficient condition for absolute convergence is that the sequence of partial sums $\sum_{k=0}^{n} |w_k|$ be bounded. The two most useful tests for absolute convergence of complex series are the ratio and root tests.

2.2.2 The Ratio Test

If $\sum w_n$ is a series of nonzero terms and if $\limsup_{n\to\infty} |\frac{w_{n+1}}{w_n}| < 1$, then the series converges absolutely. If $|\frac{w_{n+1}}{w_n}| \geq 1$ for all sufficiently large n, the series diverges.

2.2.3 The Root Test

Let $\sum w_n$ be any complex series. If $\limsup_{n\to\infty} |w_n|^{1/n} < 1$, the series converges absolutely, while if $\limsup_{n\to\infty} |w_n|^{1/n} > 1$, the series diverges.

The ratio test is usually (but not always) easier to apply in explicit examples, but the root test has a somewhat wider range of applicability and, in fact, is the test that we are going to use to obtain some basic properties of power series. Proofs and a discussion of the relative utility of the tests can be found in most texts on real analysis.

We now consider sequences and series of complex-valued functions.

2.2.4 Theorem

Let $\{f_n\}$ be sequence of complex-valued functions on a set S. Then $\{f_n\}$ *converges pointwise* on S (that is, for each $z \in S$, the sequence $\{f_n(z)\}$ is convergent in $\mathbb{C}$) iff $\{f_n\}$ is *pointwise Cauchy* (that is, for each $z \in S$, the sequence $\{f_n(z)\}$ is a Cauchy sequence in $\mathbb{C}$). Also, $\{f_n\}$ *converges uniformly* iff $\{f_n\}$ is *uniformly Cauchy* on S, in other words, $|f_n(z) - f_m(z)| \to 0$ as $m, n \to \infty$, uniformly for $z \in S$.

(The above result holds just as well if the f_n take their values in an arbitrary complete metric space.)

Proof. As in the real variables case; see Problem 2.2.1. ♣

The next result gives the most useful test for uniform convergence of infinite series of functions.

2.2.5 The Weierstrass M-Test

Let $g_1, g_2, \dots$ be complex-valued functions on a set S, and assume that $|g_n(z)| \le M_n$ for all $z \in S$. If $\sum_{n=1}^\infty M_n < +\infty$, then the series $\sum_{n=1}^\infty g_n(z)$ converges uniformly on S.

Proof. Let $f_n = \sum_{k=1}^n g_k$; it follows from the given hypothesis that $\{f_n\}$ is uniformly Cauchy on S. The result now follows from (2.2.4). ♣

We now consider *power series*, which are series of the form $\sum_{n=0}^\infty a_n(z-z_0)^n$, where z_0 and the a_n are complex numbers. Thus we are dealing with series of functions $\sum_{n=0}^\infty f_n$ of a very special type, namely $f_n(z) = a_n(z-z_0)^n$. Our first task is to describe the sets $S \subseteq \mathbb{C}$ on which such a series will converge.

2.2.6 Theorem

If $\sum_{n=0}^\infty a_n(z-z_0)^n$ converges at the point z with $|z-z_0| = r$, then the series converges absolutely on $D(z_0, r)$, uniformly on each closed subdisk of $D(z_0, r)$, hence uniformly on each compact subset of $D(z_0, r)$.

Proof. We have $|a_n(z'-z_0)^n| = |a_n(z-z_0)^n| \left|\frac{z'-z_0}{z-z_0}\right|^n$. The convergence at z implies that $a_n(z-z_0)^n \to 0$, hence the sequence $\{a_n(z-z_0)^n\}$ is bounded. If $|z'-z_0| \le r' < r$, then

$$\left|\frac{z'-z_0}{z-z_0}\right| \le \frac{r'}{r} < 1$$

proving absolute convergence at z' (by comparison with a geometric series). The Weierstrass M-test shows that the series converges uniformly on $\overline{D}(z_0, r')$. ♣

We now describe convergence in terms of the coefficients a_n.

2.2.7 Theorem

Let $\sum_{n=0}^\infty a_n(z-z_0)^n$ be a power series. Let $r = [\limsup_{n\to\infty}(|a_n|^{1/n})]^{-1}$, the radius of convergence of the series. (Adopt the convention that $1/0 = \infty, 1/\infty = 0$.) The series converges absolutely on $D(z_0, r)$, uniformly on compact subsets. The series diverges for $|z-z_0| > r$.

Proof. We have $\limsup_{n\to\infty} |a_n(z-z_0)^n|^{1/n} = (|z-z_0|)/r$, which will be less than 1 if $|z-z_0| < r$. By (2.2.3), the series converges absolutely on $D(z_0, r)$. Uniform convergence on compact subsets follows from (2.2.6). (We do not necessarily have convergence for $|z-z_0| = r$, but we do have convergence for $|z-z_0| = r'$, where $r' < r$ can be chosen arbitrarily close to r.) If the series converges at some point z with $|z-z_0| > r$, then by (2.2.6) it converges absolutely at points z' such that $r < |z'-z_0| < |z-z_0|$. But then $(|z-z_0|)/r > 1$, contradicting (2.2.3). ♣

2.2.8 Definition

Let $C(z_0, r)$ denote the circle with center z_0 and radius r. then $\int_{C(z_0,r)} f(z)\,dz$ is defined as $\int_\gamma f(z)\,dz$ where $\gamma(t) = z_0 + re^{it}, 0 \le t \le 2\pi$.

The following result provides the essential equipment needed for the theory of power series. In addition, it illustrates the striking difference between the concept of differentiability of complex functions and the analogous idea in the real case. We are going to show that if f is analytic on a closed disk, then the value of f at any interior point is completely determined by its values on the boundary, and furthermore there is an explicit formula describing the dependence.

2.2.9 Cauchy's Integral Formula for a Circle

Let f be analytic on Ω and let $D(z_0, r)$ be a disk such that $\overline{D}(z_0, r) \subseteq \Omega$. Then

$$f(z) = \frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(w)}{w-z}\,dw, \quad z \in D(z_0, r).$$

Proof. Let $D(z_0, \rho)$ be a disk such that $\overline{D}(z_0, r) \subseteq D(z_0, \rho) \subseteq \Omega$. Fix $z \in D(z_0, r)$ and define a function g on $D(z_0, \rho)$ by

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \ne z \\ f'(z) & \text{if } w = z. \end{cases}$$

Then g is continuous on $D(z_0, \rho)$ and analytic on $D(z_0, \rho) \setminus \{z\}$, so we may apply (2.1.13) to get $\int_{C(z_0,r)} g(w)\,dw = 0$. Therefore

$$\frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(w)}{w-z}\,dw = \frac{f(z)}{2\pi i}\int_{C(z_0,r)} \frac{1}{w-z}\,dw.$$

Now

$$\int_{C(z_0,r)} \frac{1}{w-z}\,dw = \int_{C(z_0,r)} \frac{1}{(w-z_0)-(z-z_0)}\,dw = \int_{C(z_0,r)} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,dw$$

The series converges uniformly on $C(z_0, r)$ by the Weierstrass M-test, and hence we may integrate term by term to obtain

$$\sum_{n=0}^{\infty}(z-z_0)^n \int_{C(z_0,r)} \frac{1}{(w-z_0)^{n+1}}\,dw.$$

But on $C(z_0, r)$ we have $w = z_0 + re^{it}, 0 \le t \le 2\pi$, so the integral on the right is, by (2.2.8),

$$\int_0^{2\pi} r^{-(n+1)} e^{-i(n+1)t} ire^{it}\,dt = \begin{cases} 0 & \text{if } n = 1, 2, \dots \\ 2\pi i & \text{if } n = 0. \end{cases}$$

We conclude that $\int_{C(z_0,r)} \frac{1}{w-z}\, dw = 2\pi i$, and the result follows. ♣

The integral appearing in Cauchy's formula is an example of what is known as an *integral of the Cauchy type*. The next result, which will be useful later, deals with these integrals.

2.2.10 Theorem

Let γ be a path (not necessarily closed) and let g be a complex-valued continuous function on γ^*. Define a function F on the open set $\Omega = \mathbb{C} \setminus \gamma^*$ by

$$F(z) = \int_\gamma \frac{g(w)}{w-z}\, dw.$$

Then F has derivatives of all orders on Ω, and

$$F^{(n)}(z) = n! \int_\gamma \frac{g(w)}{(w-z)^{n+1}}\, dw$$

for all $z \in \Omega$ and all $n = 0, 1, 2, \ldots$ (take $F^{(0)} = F$). Furthermore, $F^{(n)}(z) \to 0$ as $|z| \to \infty$.

Proof. We use an induction argument. The formula for $F^{(n)}(z)$ is valid for $n = 0$, by hypothesis. Assume that the formula holds for a given n and all $z \in \Omega$; fix $z_1 \in \Omega$ and choose $r > 0$ small enough that $D(z_1, r) \subseteq \Omega$. For any point $z \in D(z_1, r)$ with $z \neq z_1$ we have

$$\frac{F^{(n)}(z) - F^{(n)}(z_1)}{z - z_1} - (n+1)! \int_\gamma \frac{g(w)}{(w-z_1)^{n+2}}\, dw$$

$$= \frac{n!}{z-z_1} \int_\gamma \frac{(w-z_1)^{n+1} - (w-z)^{n+1}}{(w-z)^{n+1}(w-z_1)^{n+1}} g(w)\, dw - (n+1)! \int_\gamma \frac{g(w)}{(w-z_1)^{n+2}} \tag{1}$$

$$= \frac{n!}{z-z_1} \int_\gamma \frac{(z-z_1)\sum_{k=0}^{n}(w-z_1)^{n-k}(w-z)^k}{(w-z)^{n+1}(w-z_1)^{n+1}} g(w)\, dw - (n+1)! \int_\gamma \frac{g(w)}{(w-z_1)^{n+2}}\, dw \tag{2}$$

where the numerator of the first integral in (2) is obtained from that in (1) by applying the algebraic identity $a^{n+1} - b^{n+1} = (a-b)\sum_{k=0}^{n} a^{n-k}b^k$ with $a = w - z_1$ and $b = w - z$. Thus

$$\left| \frac{F^{(n)}(z) - F^{(n)}(z_1)}{z - z_1} - (n+1)! \int_\gamma \frac{g(w)}{(w-z_1)^{n+2}}\, dw \right|$$

$$= n! \left| \int_\gamma \frac{\sum_{k=0}^{n}(w-z_1)^{n-k+1}(w-z)^k - (n+1)(w-z)^{n+1}}{(w-z)^{n+1}(w-z_1)^{n+2}} g(w)\, dw \right|$$

$$\leq n!\left[\max_{w\in\gamma^*}\left|\frac{\sum_{k=0}^{n}(w-z_1)^{n-k+1}(w-z)^k-(n+1)(w-z)^{n+1}}{(w-z)^{n+1}(w-z_1)^{n+2}}g(w)\right|\right]L(\gamma)$$

by the M-L theorem. But the max that appears in brackets approaches 0 as $z \to z_1$, since $\sum_{k=0}^{n}(w-z_1)^{n-k+1}(w-z)^k \to \sum_{k=0}^{n}(w-z_1)^{n+1} = (n+1)(w-z_1)^{n+1}$. Hence

$$\frac{F^{(n)}(z)-F^{(n)}(z_1)}{z-z_1} \to (n+1)!\int_\gamma \frac{g(w)}{(w-z_1)^{n+2}}\,dw$$

as $z \to z_1$, and the statement of the theorem follows by induction. The fact that $|F^{(n)}(z)| \to 0$ as $|z| \to \infty$ is a consequence of the M-L theorem; specifically,

$$F^{(n)}(z)| \leq n!\left[\max_{w\in\gamma^*}\frac{|g(w)|}{|w-z|^{n+1}}\right]L(\gamma). \ ♣$$

Theorems 2.2.9 and 2.2.10 now yield some useful corollaries.

2.2.11 Corollary

If f is analytic on Ω, then f has derivatives of all orders on Ω. Moreover, if $\overline{D}(z_0,r) \subseteq \Omega$, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C(z_0,r)}\frac{f(w)}{(w-z)^{n+1}}\,dw, \quad z \in D(z_0,r).$$

Proof. Apply (2.2.10) to the Cauchy integral formula (2.2.9). ♣

2.2.12 Corollary

If f has a primitive on Ω, then f is analytic on Ω.

Proof. Apply (2.2.11) to any primitive for f. ♣

2.2.13 Corollary

If f is continuous on Ω and analytic on $\Omega \setminus \{z_0\}$, then f is analytic on Ω.

Proof. Choose any disk D such that $D \subseteq \Omega$. By (2.1.13), f has a primitive on D, hence by (2.2.12), f is analytic on D. It follows that f is analytic on Ω. ♣

The next result is a converse to Cauchy's theorem for triangles.

2.2.14 Morera's Theorem

Suppose f is continuous on Ω and $\int_T f(z)\,dz = 0$ for each triangle T such that $\hat{T} \subseteq \Omega$. Then f is analytic on Ω.

Proof. Let D be any disk contained in Ω. The hypothesis implies that f has a primitive on D [see the discussion preceding (2.1.8)]. Thus by (2.2.12), f is analytic on D. Since D is an arbitrary disk in Ω, f is analytic on Ω. ♣

One of many applications of Morera's theorem is the Schwarz reflection principle, which deals with the problem of extending an analytic function to a larger domain.

2.2.15 The Schwarz Reflection Principle

Suppose that f is analytic on the open upper half plane $\mathbb{C}^+ = \{z : \operatorname{Im} z > 0\}$, f is continuous on the closure $\mathbb{C}^+ \cup \mathbb{R}$ of $\mathbb{C}^+$, and $\operatorname{Im} f(z) = 0$ for $z \in \mathbb{R}$. Then f has an analytic extension to all of $\mathbb{C}$.

Proof. We will give an outline of the argument, leaving the details to the problems at the end of the section. Extend f to a function f^* defined on $\mathbb{C}$ by

$$f^*(z) = \begin{cases} f(z), & z \in \mathbb{C}^+ \cup \mathbb{R} \\ \\ \overline{f(\overline{z})}, & z \notin \mathbb{C}^+ \cup \mathbb{R}. \end{cases}$$

Then f^* is analytic on $\mathbb{C} \setminus \mathbb{R}$ and continuous on $\mathbb{C}$ (Problem 10). One can then use Morera's theorem to show that f^* is analytic on $\mathbb{C}$ (Problem 11). ♣

We now complete the discussion of the connection between analytic functions and power series, showing in essence that the two notions are equivalent. We say that a function $f : \Omega \to \mathbb{C}$ is *representable in* Ω *by power series* if given $D(z_0, r) \subseteq \Omega$, there is a sequence $\{a_n\}$ of complex numbers such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $z \in D(z_0, r)$.

2.2.16 Theorem

If f is analytic on Ω, then f is representable in Ω by power series. In fact, if $D(z_0, r) \subseteq \Omega$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \quad z \in D(z_0, r).$$

As is the usual practice, we will call this series the *Taylor expansion* of f about z_0.

Proof. Let $D(z_0, r) \subseteq \Omega$, fix any $z \in D(z_0, r)$, and choose r_1 such that $|z - z_0| < r_1 < r$. By (2.2.9), Cauchy's formula for a circle,

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0, r_1)} \frac{f(w)}{w - z} \, dw.$$

Now for $w \in C(z_0, r_1)$,

$$\frac{f(w)}{w - z} = \frac{f(w)}{(w - z_0) - (z - z_0)} = \frac{f(w)}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \sum_{n=0}^{\infty} f(w) \frac{(z - z_0)^n}{(w - z_0)^{n+1}}.$$

The n-th term of the series has absolute value at most

$$\max_{w \in C(z_0, r_1)} |f(w)| \cdot \frac{|z - z_0|^n}{r_1^{n+1}} = \frac{1}{r_1} \max_{w \in C(z_0, r_1)} |f(w)| \left[\frac{|z - z_0|}{r_1} \right]^n.$$

Since $\frac{|z - z_0|}{r_1} < 1$, the Weierstrass M-test shows that the series converges uniformly on $C(z_0, r_1)$. Hence we may integrate term by term, obtaining

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{C(z_0, r_1)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw \right] (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

by (2.2.11). ♣

In order to prove the converse of (2.2.16), namely that a function representable in Ω by power series is analytic on Ω, we need the following basic result.

2.2.17 Theorem

Let $\{f_n\}$ be a sequence of analytic functions on Ω such that $f_n \to f$ uniformly on compact subsets of Ω. Then f is analytic on Ω, and furthermore, $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of Ω for each $k = 1, 2, \ldots$.

Proof. First let $\overline{D}(z_0, r)$ be any closed disk contained in Ω. Then we can choose $\rho > r$ such that $\overline{D}(z_0, \rho) \subseteq \Omega$ also. For each $z \in D(z_0, \rho)$ and $n = 1, 2, \ldots$, we have, by (2.2.9),

$$f_n(z) = \frac{1}{2\pi i} \int_{C(z_0,\rho)} \frac{f(w)}{w - z}\, dw.$$

By (2.2.10), f is analytic on $D(z_0, \rho)$. It follows that f is analytic on Ω. Now by (2.2.11),

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{C(z_0,\rho)} \frac{f_n(w) - f(w)}{(w - z)^{k+1}}\, dw$$

and if z is restricted to $\overline{D}(z_0, r)$, then by the M-L theorem,

$$|f_n^{(k)}(z) - f^{(k)}(z)| \le \frac{k!}{2\pi} \left[\max_{w \in C(z_0,\rho)} |f_n(w) - f(w)| \right] \frac{2\pi\rho}{(\rho - r)^{k+1}} \to 0 \text{ as } n \to \infty.$$

Thus we have shown that f is analytic on Ω and that $f_n^{(k)} \to f^{(k)}$ uniformly on closed *subdisks* of Ω. Since any compact subset of Ω can be covered by finitely many closed subdisks, the statement of the theorem follows. ♣

The converse of (2.2.16) can now be readily obtained.

2.2.18 Theorem

If f is representable in Ω by power series, then f is analytic on Ω.

Proof. Let $D(z_0, r) \subseteq \Omega$, and let $\{a_n\}$ be such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, z \in D(z_0, r)$. By (2.2.7), the series converges uniformly on compact subsets of $D(z_0, r)$, hence by (2.2.17), f is analytic on Ω. ♣

Remark

Since the above series converges uniformly on compact subsets of $D(z_0, r)$, Theorem 2.2.17 also allows us to derive the power series expansion of $f^{(k)}$ from that of f, and to show that the coefficients $\{a_n\}$ are uniquely determined by z_0 and f. For if $f(z)$ is given by $\sum_{n=0}^{\infty} a_n (z - z_0)^n, z \in D(z_0, r)$, we may differentiate term by term to obtain

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k},$$

and if we set $z = z_0$, we find that

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

We conclude this section with a result promised in Chapter 1 [see (1.6.1)].

2.2.19 Theorem

If $f = u + iv$ is analytic on Ω, then u and v are harmonic on Ω.

Proof. By (1.4.2), $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$. But by (2.2.11), f' is also analytic on Ω, and thus the Cauchy-Riemann equations for f' are also satisfied. Consequently,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right).$$

These partials are all continuous because f'' is also analytic on Ω. ♣

Problems

1. Prove Theorem 2.2.4.
2. If $\sum_n z^n$ has radius of convergence r, show that the differentiated series $\sum na_n z^{n-1}$ also has radius of convergence r.
3. Let $f(x) = e^{-1/x^2}, x \neq 0; f(0) = 0$. Show that f is infinitely differentiable on $(-\infty, \infty)$ and $f^{(n)}(0) = 0$ for all n. Thus the Taylor series for f is identically 0, hence does not converge to f. Conclude that if $r > 0$, there is no function g analytic on $D(0, r)$ such that $g = f$ on $(-r, r)$.
4. Let $\{a_n : n = 0, 1, 2 \dots\}$ be an arbitrary sequence of complex numbers.
(a) If $\limsup_{n\to\infty} |a_{n+1}/a_n| = \alpha$, what conclusions can be drawn about the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$?
(b) If $|a_{n+1}/a_n|$ approaches a limit α, what conclusions can be drawn?
5. If f is analytic at z_0, show that it is not possible that $|f^{(n)}(z_0)| > n!b_n$ for all $n = 1, 2, \dots,$ where $(b_n)^{1/n} \to \infty$ as $n \to \infty$.
6. Let $R_n(z)$ be the remainder after the term of degree n in the Taylor expansion of a function f about z_0.
(a) Show that

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z)(w - z_0)^{n+1}}\, dw,$$

where $\Gamma = C(z_0, r_1)$ as in (2.2.16).
(b) If $|z - z_0| \le s < r_1$, show that

$$|R_n(z)| \le A(s/r_1)^{n+1}, \text{ where } A = M_f(\Gamma)r_1/(r_1 - s)$$

and $M_f(\Gamma) = \max\{|f(w)| : w \in \Gamma\}$.

7. (Summation by parts). Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers. If $\Delta b_k = b_{k+1} - b_k$, show that

$$\sum_{k=r}^{s} a_k \Delta b_k = a_{s+1}b_{s+1} - a_r b_r - \sum_{k=r}^{s} b_{k+1}\Delta a_k.$$

8. (a) If $\{b_n\}$ is bounded and the a_n are real and greater than 0, with $a_1 \geq a_2 \geq \cdots \to 0$, show that $\sum_{n=1}^{\infty} a_n \Delta b_n$ converges.
(b) If $b_n = b_n(z)$, that is, the b_n are *functions* from a set S to $\mathbb{C}$, the b_n are uniformly bounded on S, and the a_n are real and decrease to 0 as in (a), show that $\sum_{n=1}^{\infty} a_n(b_{n+1}(z) - b_n(z))$ converges uniformly on S.
9. (a) Show that $\sum_{n=1}^{\infty} z^n/n$ converges when $|z| = 1$, except at the single point $z = 1$.
(b) Show that $\sum_{n=1}^{\infty} (\sin nx)/n$ converges for real x, uniformly on
$\{x : 2k\pi + \delta \leq x \leq 2(k+1)\pi - \delta\}$, $\delta > 0$, k an integer.
(c) Show that $\sum_{n=1}^{\infty} (\sin nz)/n$ diverges if x is not real. (The complex sine function will be discussed in the next chapter. It is defined by $\sin w = (e^{iw} - e^{-iw})/2i$.
10. Show that the function f^* occurring in the proof of the Schwarz reflection principle is analytic on $\mathbb{C} \setminus \mathbb{R}$ and continuous on $\mathbb{C}$.
11. Show that f^* is analytic on $\mathbb{C}$.
12. Use the following outline to give an alternative proof of the Cauchy integral formula for a circle.
(a) Let

$$F(z) = \int_{C(z_0,r)} \frac{1}{w - z}\, dw, \ z \notin C(z_0, r).$$

Use (2.2.10), (2.1.6) and (2.1.7b) to show that F is constant on $D(z_0, r)$.
(b) $F(z_0) = 2\pi i$ by direct computation.
Theorem 2.2.9 now follows, thus avoiding the series expansion argument that appears in the text.
13. (a) Suppose f is analytic on $D(a, r)$. Prove that for $0 \leq r < R$,

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi r^n} \int_{-\pi}^{\pi} |f(a + re^{it})|\, dt.$$

(b) Prove that if f is an entire function such that for some $M > 0$ and some natural number k, $|f(z)| \leq M|z|^k$ for $|z|$ sufficiently large, then f is a polynomial of degree at most k.
(c) Let f be an entire function such that $|f(z)| \leq 1 + |z|^{3/2}$ for all z. Prove that there are complex numbers a_0, a_1 such that $f(z) = a_0 + a_1 z$.
14. Let $\{a_n : n = 0, 1, \ldots\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} |a_n| < \infty$ but $\sum_{n=0}^{\infty} n|a_n| = \infty$. Prove that the radius of convergence of the power series $\sum a_n z^n$ is equal to 1.
15. Let $\{f_n\}$ be a sequence of analytic functions on Ω such that $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Give a proof that f is analytic on Ω, based on

Morera's theorem [rather than (2.2.10), which was the main ingredient in the proof of (2.2.17)]. Note that in the present problem we need not prove that $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of Ω.

2.3 The Exponential and Complex Trigonometric Functions

In this section, we use our results on power series to complete the discussion of the exponential function and to introduce some of the other elementary functions.

Recall (Section 1.5) that exp is defined on $\mathbb{C}$ by $\exp(x+iy) = e^x(\cos y + i \sin y)$; thus exp has magnitude e^x and argument y. The function exp satisfies a long list of properties; for the reader's convenience, we give the justification of each item immediately after the statement.

2.3.1 Theorem

(a) exp is an entire function [this was proved in (1.5.2)].

(b) $\exp(z) = \sum_{n=0}^{\infty} z^n/n!, z \in \mathbb{C}$.

Apply (a) and (2.2.16), using the fact [see (1.5.2)] that exp is its own derivative.

(c) $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$.

Fix $z_0 \in \mathbb{C}$; for each $z \in \mathbb{C}$, we have, by (2.2.16),

$$\exp(z) = \sum_{n=0}^{\infty} \frac{\exp(z_0)}{n!}(z-z_0)^n = \exp(z_0)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!} = \exp(z_0)\exp(z-z_0) \text{ by (b).}$$

Now set $z_0 = z_1$ and $z = z_1 + z_2$.

(d) exp has no zeros in $\mathbb{C}$.

By (c), $\exp(z-z) = \exp(z)\exp(-z)$. But $\exp(z-z) = \exp(0) = 1$, hence $\exp(z) \neq 0$.

(e) $\exp(-z) = 1/\exp(z)$ (the argument of (d) proves this also).

(f) $\exp(z) = 1$ iff z is an integer multiple of $2\pi i$.

$\exp(x+iy) = 1$ iff $e^x \cos y = 1$ and $e^x \sin y = 0$ iff $e^x \cos y = 1$ and $\sin y = 0$ iff $x = 0$ and $y = 2n\pi$ for some n.

(g) $|\exp(z)| = e^{\operatorname{Re} z}$ (by definition of exp).

(h) exp has $2\pi i$ as a period, and any other period is an integer multiple of $2\pi i$.

$\exp(z+w) = \exp(z)$ iff $\exp(w) = 1$ by (c), and the result follows from (f).

(i) exp maps an arbitrary vertical line $\{z : \operatorname{Re} z = x_0\}$ onto the circle with center 0 and radius e^{x_0}, and exp maps an arbitrary horizontal line $\{z : \operatorname{Im} z = y_0\}$ one-to-one onto the open ray from 0 through $\exp(iy_0)$.

$\{\exp(z) : \operatorname{Re} z = x_0\} = \{e^{x_0}(\cos y + i \sin y) : y \in \mathbb{R}\}$, which is the circle with center 0 and radius e^{x_0} (covered infinitely many times). Similarly, we have $\{\exp(z) : \operatorname{Im} z = y_0\} = \{e^x e^{iy_0} : x \in \mathbb{R}\}$, which is the desired ray.

(j) For each real number α, exp restricted to the horizontal strip $\{x+iy : \alpha \le y < \alpha+2\pi\}$, is a one-to-one map onto $\mathbb{C} \setminus \{0\}$.

This follows from (i) and the observation that as y_0 ranges over $[\alpha, \alpha+2\pi)$, the open rays from 0 through e^{iy_0} sweep out $\mathbb{C} \setminus \{0\}$. ♣

Notation

We will often write e^z for $\exp(z)$. We now define $\sin z$ and $\cos z$ by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

These definitions are consistent with, and are motivated by, the fact that $e^{iy} = \cos y + i \sin y, y \in \mathbb{R}$.

Since exp is an entire function, it follows from the chain rule that sin and cos are also entire functions and the usual formulas $\sin' = \cos$ and $\cos' = -\sin$ hold. Also, it follows from property (f) of exp that sin and cos have no additional zeros in the complex plane, other than those on the real line. (Note that $\sin z = 0$ iff $e^{iz} = e^{-iz}$ iff $e^{2iz} = 1$.) However, unlike $\sin z$ and $\cos z$ for real z, sin and cos are *not* bounded functions. This can be deduced directly from the above definitions, or from Liouville's theorem, to be proved in the next section.

The familiar power series representations of sin and cos hold [and may be derived using (2.2.16)]:

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Other standard trigonometric functions can be defined in the usual way; for example, $\tan z = \sin z / \cos z$. Usual trigonometric identities and differentiation formulas hold, for instance, $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$, $\frac{d}{dz} \tan z = \sec^2 z$, and so on.

Hyperbolic functions are defined by

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

The following identities can be derived from the definitions:

$$\cos iz = \cosh z, \quad \sin iz = i \sinh z$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \quad \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

Also, $\sinh z = 0$ iff $z = in\pi$, n an integer; $\cosh z = 0$ iff $z = i(2n+1)\pi/2$, n an integer.

Problems

1. Show that for any integer k, $\sin z$ maps the strip $\{x+iy : (2k-1)\pi/2 < x < (2k+1)\pi/2\}$ one-to-one onto $\mathbb{C} \setminus \{u+iv : v = 0, |u| \geq 1\}$, and maps $\{x+iy : x = (2k+1)\pi/2, y \geq 0\} \cup \{x+iy : x = (2k-1)\pi/2, y \leq 0\}$ one-to-one onto $\{u+iv : v = 0, |u| \geq 1\}$.
2. Find all solutions of the equation $\sin z = 3$.
3. Calculate $\int_{C(0,1)} \frac{\sin z}{z^4}\, dz$.
4. Prove that given $r > 0$, there exists n_0 such that if $n \geq n_0$, then $1+z+z^2/2!+\cdots+z^n/n!$ has all its zeros in $|z| > r$.
5. Let f be an entire function such that $f'' + f = 0$, $f(0) = 0$, and $f'(0) = 1$. Prove that $f(z) = \sin z$ for all $z \in \mathbb{C}$.
6. Let f be an entire function such that $f' = f$ and $f(0) = 1$. What follows and why?

2.4 Further Applications

In this section, we apply the preceding results in a variety of ways. The first two of these are consequences of the Cauchy integral formula for derivatives (2.2.11).

2.4.1 Cauchy's Estimate

Let f be analytic on Ω, and let $\overline{D}(z_0, r) \subseteq \Omega$. Then

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{z \in C(z_0,r)} |f(z)|.$$

Proof. This is immediate from (2.2.11) and the M-L theorem. ♣

Remark

If $f(z) = z^n$ and $z_0 = 0$, we have $f^{(n)}(z_0) = n! = (n!/r^n) \max_{z\in C(z_0,r)} |f(z)|$, so the above inequality is sharp.

2.4.2 Liouville's Theorem

If f is a bounded entire function, then f is constant.

Proof. Assume that $|f(z)| \leq M < \infty$ for all $z \in \mathbb{C}$, and fix $z_0 \in \mathbb{C}$. By (2.4.1), $|f'(z_0)| \leq M/r$ for all $r > 0$. Let $r \to \infty$ to conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f' \equiv 0$, hence f is constant on $\mathbb{C}$ by (2.1.7b). ♣

2.4.3 The Fundamental Theorem of Algebra

Suppose $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is polynomial of degree $n \geq 1$. Then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Proof. Since

$$|P(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \geq |z|^n \left| \frac{a_n}{2} \right|$$

for all sufficiently large $|z|$, it follows that $|P(z)| \to \infty$ as $z| \to \infty$. If $P(z)$ is never 0, then $1/P$ is an entire function. Moreover, $|1/P(z)| \to 0$ as $|z| \to \infty$, and therefore $1/P$ is bounded. By (2.4.2), $1/P$ is constant, contradicting $\deg P \geq 1$. ♣

Recall that if P is a polynomial of degree $n \geq 1$ and $P(z_0) = 0$, we may write $P(z) = (z - z_0)^m Q(z)$ where m is a positive integer and $Q(z)$ is a polynomial (possibly constant) such that $Q(z_0) \neq 0$. In this case P is said to have a zero of order m at z_0. The next definition extends the notion of the order of a zero to analytic functions in general.

2.4.4 Definition

Let f be analytic on Ω and $z_0 \in \Omega$. We say that f *has a zero of order* m *at* z_0 if there is an analytic function g on Ω such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in \Omega$.

2.4.5 Remark

In terms of the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, f has a zero of order m at z_0 iff $a_0 = a_1 = \cdots = a_{m-1} = 0$, while $a_m \neq 0$. Equivalently, $f^{(n)}(z_0) = 0$ for $n = 0, \ldots, m-1$, while $f^{(m)}(z_0) \neq 0$ (see Problem 2).

2.4.6 Definition

If $f : \Omega \to \mathbb{C}$, the *zero set* of f is defined as $Z(f) = \{z \in \Omega : f(z) = 0\}$.

Our next major result, the identity theorem for analytic functions, is a consequence of a topological property of $Z(f)$.

2.4.7 Lemma

Let f be analytic on Ω, and let L be the set of limit points (also called accumulation points or cluster points) of $Z(f)$ in Ω. Then L is both open and closed in Ω.

Proof. First note that $L \subseteq Z(f)$ by continuity of f. Also, L is closed in Ω because the set of limit points of any subset of Ω is closed in Ω. (If $\{z_n\}$ is a sequence in L such that $z_n \to z$, then given $r > 0$, $z_n \in D(z, r)$ for n sufficiently large. Since z_n is a limit point of $Z(f)$, $D(z, r)$ contains infinitely many points of $Z(f)$ different from z_n, and hence infinitely many points of $Z(f)$ different from z. Thus $z \in L$ also.) It remains to show that L is open in Ω. Let $z_0 \in L$, and write $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, z \in D(z_0, r) \subseteq \Omega$. Now $f(z_0) = 0$, and hence either f has a zero of order m at z_0 (for some m), or else $a_n = 0$ for all n. In the former case, there is a function g analytic on Ω such that $f(z) = (z - z_0)^m g(z), z \in \Omega$, with $g(z_0) \neq 0$. By continuity of g, $g(z) \neq 0$ for all z sufficiently close to z_0, and consequently z_0 is an isolated point of $Z(f)$. But then $z_0 \notin L$, contradicting out assumption. Thus, it must be the case that $a_n = 0$ for all n, so that $f \equiv 0$ on $D(z_0, r)$. Consequently, $D(z_0, r) \subseteq L$, proving that L is open in Ω. ♣

2.4.8 The Identity Theorem

Suppose f is analytic on the open connected set Ω. Then either f is identically zero on Ω or else $Z(f)$ has no limit point in Ω. Equivalently, if $Z(f)$ has a limit point in Ω, then f is identically 0 on Ω.

Proof. By (2.4.7), the set L of limit points of $Z(f)$ is both open and closed in Ω. Since Ω is connected, either $L = \Omega$, in which case $f \equiv 0$ on Ω, or $L = \emptyset$, so that $Z(f)$ has no limit point in Ω. ♣

2.4.9 Corollary

If f and g are analytic on Ω and $\{z \in \Omega : f(z) = g(z)\}$ has a limit point in Ω, then $f \equiv g$.

Proof. Apply the identity theorem to $f - g$. ♣

Our next application will be to show (roughly) that the absolute value of a function analytic on a set S cannot attain a maximum at an interior point of S. As a preliminary we show that the value of an analytic function at the center of a circle is the average of its values on the circumference.

2.4.10 Theorem

Suppose f is analytic on Ω and $\overline{D}(z_0, r) \subseteq \Omega$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Proof. Use (2.2.9), Cauchy's integral formula for a circle, with $z = z_0$. ♣

The other preliminary to the proof of the maximum principle is the following fact about integrals.

2.4.11 Lemma

Suppose $\varphi : [a, b] \to \mathbb{R}$ is continuous, $\varphi(t) \leq k$ for all t, while the average of φ, namely $\frac{1}{b-a} \int_a^b \varphi(t)\, dt$, is at least k. Then $\varphi(t) = k$ for all t.

Proof. Observe that

$$0 \leq \int_a^b [k - \varphi(t)]\, dt = k(b - a) - \int_a^b \varphi(t)\, dt \leq 0. \ ♣$$

We now consider the maximum principle, which is actually a collection of closely related results rather than a single theorem. We will prove four versions of the principle, arranged in order of decreasing strength.

2.4.12 Maximum Principle

Let f be analytic on the open connected set Ω.

(a) If $|f|$ assumes a local maximum at some point in Ω, then f is constant on Ω.

(b) If $\lambda = \sup\{|f(z)| : z \in \Omega\}$, then either $|f(z)| < \lambda$ for all $z \in \Omega$ or f is constant on Ω.

(c) If Ω is a bounded region and $M \geq 0$ is such that $\limsup_{n\to\infty} |f(z_n)| \leq M$ for each sequence $\{z_n\}$ in Ω that converges to a boundary point of Ω, then $|f(z)| < M$ for all $z \in \Omega$ or f is constant on Ω.

(d) Let Ω be a bounded region, with f continuous on the closure $\overline{\Omega}$ of Ω. Denote the boundary of Ω by $\partial\Omega$, and let $M_0 = \max\{|f(z)| : z \in \partial\Omega\}$. Then either $|f(z)| < M_0$ for all $z \in \Omega$ or f is constant on Ω. Consequently, $\max\{|f(z)| : z \in \overline{\Omega}\} = \max\{|f(z)| : z \in \partial\Omega\}$.

Proof.

(a) If $|f|$ assumes a local maximum at $z_0 \in \Omega$, then for some $\delta > 0$, $|f(z)| \leq |f(z_0)|$ for $|z - z_0| < \delta$. If $f(z_0) = 0$, then $f(z) = 0$ for all $z \in D(z_0, \delta)$, so $f \equiv 0$ by the identity theorem. So assume that $f(z_0) \neq 0$. If $0 < r < \delta$, then (2.4.10) with both sides divided by $f(z_0)$ yields

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{f(z_0)}\,dt.$$

Taking the magnitude of both sides, we obtain

$$1 \leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f(z_0 + re^{it})}{f(z_0)}\right| dt \leq 1$$

because $|f(z_0 + re^{it})| \leq |f(z_0)|$ for all $t \in [0, 2\pi]$. Since this holds for all $r \in (0, \delta)$, the preceding lemma (2.4.11) gives $|f(z)/f(z_0)| = 1, z \in D(z_0, \delta)$. Now take the real part (rather than the magnitude) of both sides of the above integral, and use the fact that for any complex number w, we have $|\operatorname{Re} w| \leq |w|$. We conclude that $\operatorname{Re}(f(z)/f(z_0)) = 1$ on $D(z_0, \delta)$. But if $|w| = \operatorname{Re} w = c$, then $w = c$, hence $f(z) = f(z_0)$ on $D(z_0, \delta)$. By the identity theorem, f is constant on Ω.

(b) If $\lambda = +\infty$ there is nothing to prove, so assume $\lambda < +\infty$. If $|f(z_0)| = \lambda$ for some $z_0 \in \Omega$, then f is constant on Ω by (a).

(c) If λ is defined as in (b), then there is a sequence $\{z_n\}$ in Ω such that $|f(z_n)| \to \lambda$. But since Ω is *bounded*, there is a subsequence $\{z_{n_j}\}$ that converges to a limit z_0. If $z_0 \in \Omega$, then $|f(z_0)| = \lambda$, hence f is constant by (b). On the other hand, if z_0 belongs to the boundary of Ω, then $\lambda \leq M$ by hypothesis. Again by (b), either $|f(z)| < \lambda \leq M$ for all $z \in \Omega$ or f is constant on Ω.

(d) Let $\{z_n\}$ be any sequence in Ω converging to a point $z_0 \in \partial\Omega$. Then $|f(z_n)| \to |f(z_0)| \leq M_0$. By (c), $|f| < M_0$ on Ω or f is constant on Ω. In either case, the maximum of $|f|$ on $\overline{\Omega}$ is equal to the maximum of $|f|$ on $\partial\Omega$. ♣

The absolute value of an analytic function may attain its minimum modulus on an open connected set without being constant (consider $f(z) = z$ on $\mathbb{C}$). However, if the function is never zero, we do have a minimum principle.

2.4.13 Minimum Principle

Let f be analytic and never 0 on the region Ω.

(a) If $|f|$ assumes a local minimum at some point in Ω, then f is constant on Ω.

(b) Let $\mu = \inf\{|f(z)| : z \in \Omega\}$; then either $|f(z)| > \mu$ for all $z \in \Omega$ or f is constant on Ω.

(c) If Ω is a bounded region and $m \geq 0$ is such that $\liminf_{n\to\infty} |f(z_n)| \geq m$ for each sequence $\{z_n\}$ that converges to a boundary point of Ω, then $|f(z)| > m$ for all $z \in \Omega$ or f is constant on Ω.

(d) Let Ω be a bounded region, with f is continuous on $\overline{\Omega}$ and $m_0 = \min\{|f(z)| : z \in \partial\Omega\}$. Then either $|f(z)| > m_0$ for all $z \in \Omega$ or f is constant on Ω. As a consequence, we have $\min\{|f(z)| : z \in \overline{\Omega}\} = \min\{|f(z)| : z \in \partial\Omega\}$.

Proof. Apply the maximum principle to $1/f$. ♣

Suppose f is analytic on the region Ω, and we put $g = e^f$. Then $|g| = e^{\operatorname{Re} f}$, and hence $|g|$ assumes a local maximum at $z_0 \in \Omega$ iff $\operatorname{Re} f$ has a local maximum at z_0. A similar statement holds for a local minimum. Furthermore, by (2.1.7b), f is constant iff $f' \equiv 0$ iff $f'e^f \equiv 0$ iff $g' \equiv 0$ iff g is constant on Ω. Thus $\operatorname{Re} f$ satisfies part (a) of both the maximum and minimum principles (note that $|g|$ is never 0). A similar argument can be given for $\operatorname{Im} f$ (put $g = e^{-if}$). Since the real and imaginary parts of an analytic function are, in particular, harmonic functions [see (2.2.19)], the question arises as to whether the maximum and minimum principles are valid for harmonic functions in general. The answer is yes, as we now proceed to show. We will need to establish one preliminary result which is a weak version of the identity theorem (2.4.8) for harmonic functions.

2.4.14 Identity Theorem for Harmonic Functions

If u is harmonic on the region Ω, and u restricted to some subdisk of Ω is constant, then u is constant on Ω.

Proof. Let $A = \{a \in \Omega : u \text{ is constant on some disk with center at } a\}$. It follows from the definition of A that A is an open subset of Ω. But $\Omega \setminus A$ is also open; to see this, let $z_0 \in \Omega \setminus A$ and $D(z_0, r) \subseteq \Omega$. By (1.6.2), u has a harmonic conjugate v on $D(z_0, r)$, so that u is the real part of an analytic function on $D(z_0, r)$. If u is constant on any subdisk of $D(z_0, r)$, then [since u satisfies (a) of the maximum (or minimum) principle, as indicated in the remarks following (2.4.13)] u is constant on $D(z_0, r)$, contradicting $z_0 \in \Omega \setminus A$. Thus $D(z_0, r) \subseteq \Omega \setminus A$, proving that $\Omega \setminus A$ is also open. Since Ω is connected and $A \neq \emptyset$ by hypothesis, we have $A = \Omega$.

Finally, fix $z_1 \in \Omega$ and let $B = \{z \in \Omega : u(z) = u(z_1)\}$. By continuity of u, B is closed in Ω, and since $A = \Omega$, B is also open in Ω. But B is not empty (it contains z_1), hence $B = \Omega$, proving that u is constant on Ω. ♣

2.4.15 Maximum and Minimum Principle for Harmonic Functions

If u is harmonic on a region Ω and u has either a local maximum or a local minimum at some point of Ω, then u is constant on Ω.

Proof. Say u has a local minimum at $z_0 \in \Omega$ (the argument for a maximum is similar). Then for some $r > 0$ we have $D(z_0, r) \subseteq \Omega$ and $u(z) \geq u(z_0)$ on $D(z_0, r)$. By (1.6.2) again, u is the real part of an analytic function on $D(z_0, r)$, and we may invoke the minimum principle [as we did in proving (2.4.14)] to conclude that u is constant on $D(z_0, r)$ and hence by (2.4.14), constant on Ω. ♣

Remark

The proof of (2.4.12) shows that part (a) of the maximum principle implies part (b), (b) implies (c), and (c) implies (d), and similarly for the minimum principle. Thus harmonic functions satisfy statements (b), (c) and (d) of the maximum and minimum principles.

We conclude this chapter with one of the most important applications of the maximum principle.

2.4.16 Schwarz's Lemma

Let f be analytic on the unit disk $D = D(0, 1)$, and assume that $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in D$. Then (a) $|f(z)| \le |z|$ on D, and (b) $|f'(0)| \le 1$. Furthermore, if equality holds in (a) for some $z \neq 0$, or if equality holds in (b), then f is a rotation of D. That is, there is a constant λ with $|\lambda| = 1$ such that $f(z) = \lambda z$ for all $z \in D$.

Proof. Define

$$g(z) = \begin{cases} f(z)/z & \text{if } z \in D \setminus \{0\} \\ f'(0) & \text{if } z = 0. \end{cases}$$

By (2.2.13), g is analytic on D. We claim that $|g(z)| \le 1$. For if $|z| < r < 1$, part (d) of the maximum principle yields

$$|g(z)| \le \max\{|g(w)| : |w| = r\} \le \frac{1}{r} \sup\{|f(w)| : w \in D\} \le \frac{1}{r}.$$

Since r may be chosen arbitrarily close to 1, we have $|g| \le 1$ on D, proving both (a) and (b). If equality holds in (a) for some $z \neq 0$, or if equality holds in (b), then g assumes its maximum modulus at a point of D, and hence g is a constant λ on D (necessarily $|\lambda| = 1$). Thus $f(z) = \lambda z$ for all $z \in D$. ♣

Schwarz's lemma will be generalized and applied in Chapter 4 (see also Problem 24).

Problems

1. Give an example of a nonconstant analytic function f on a region Ω such that f has a limit point of zeros at a point outside of Ω.
2. Verify the statements made in (2.4.5).
3. Consider the four forms of the maximum principle (2.4.12), for continuous rather than analytic functions. What can be said about the relative strengths of the statements? The proof in the text shows that (a) implies (b) implies (c) implies (d), but for example, does (b) imply (a)? (The region Ω is assumed to be one particular fixed open connected set, that is, the statement of the theorem does not have "for all Ω" in it.)
4. (L'Hospital's rule). Let f and g be analytic at z_0, and not identically zero in any neighborhood of z_0. If $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} g(z) = 0$, show that $f(z)/g(z)$ approaches a limit (possibly ∞) as $z \to z_0$, and $\lim_{z\to z_0} f(z)/g(z) = \lim_{z\to z_0} f'(z)/g'(z)$.
5. If f is analytic on a region Ω and $|f|$ is constant on Ω, show that f is constant on Ω.

6. Let f be continuous on the closed unit disk $\overline{D}$, analytic on D, and real-valued on ∂D. Prove that f is constant.
7. Let $f(z) = \sin z$. Find $\max\{|f(z)| : z \in K\}$ where $K = \{x + iy : 0 \le x, y \le 2\pi\}$.
8. (A generalization of part (d) of the maximum principle). Suppose K is compact, f is continuous on K, and f is analytic on K°, the interior of K. Show that
$$\max_{z\in K} |f(z)| = \max_{z\in\partial K} |f(z)|.$$
Moreover, if $|f(z_0)| = \max_{z\in K} |f(z)|$ for some $z_0 \in K^\circ$, then f is constant on the component of K° that contains z_0.
9. Suppose that Ω is a bounded open set (not necessarily connected), f is continuous on $\overline{\Omega}$ and analytic on Ω. Show that $\max\{|f(z)| : z \in \overline{\Omega}\} = \max\{|f(z)| : z \in \partial\Omega\}$.
10. Give an example of a nonconstant harmonic function u on $\mathbb{C}$ such that $u(z) = 0$ for each real z. Thus the disk that appears in the statement of Theorem 2.4.14 cannot be replaced by just any subset of $\mathbb{C}$ having a limit point in $\mathbb{C}$.
11. Prove that an open set Ω is connected iff for all f, g analytic on Ω, the following holds: If $f(z)g(z) = 0$ for every $z \in \Omega$, then either f or g is identically zero on Ω. (This says that the ring of analytic functions on Ω is an integral domain iff Ω is connected.)
12. Suppose that f is analytic on $\mathbb{C}^+ = \{z : \operatorname{Im} z > 0\}$ and continuous on $S = \mathbb{C}^+ \cup (0, 1)$. Assume that $f(x) = x^4 - 2x^2$ for all $x \in (0, 1)$. Show that $f(i) = 3$.
13. Let f be an entire function such that $|f(z)| \ge 1$ for all z. Prove that f is constant.
14. Does there exist an entire function f, not identically zero, for which $f(z) = 0$ for every z in an uncountable set of complex numbers?
15. Explain why knowing that the trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ for all *real* α and β implies that the same identity holds for all *complex* α and β.
16. Suppose f is an entire function and $\operatorname{Im}(f(z)) \ge 0$ for all z. Prove that f is constant. (Consider $\exp(if)$.)
17. Suppose f and g are analytic and nonzero on $D(0, 1)$, and $\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)}, n = 2, 3, \dots$. Prove that f/g is constant on $D(0, 1)$.
18. Suppose that f is an entire function, $f(0) = 0$ and $|f(z) - e^z \sin z| < 4$ for all z. Find a formula for $f(z)$.
19. Let f and g be analytic on $D = D(0, 1)$ and continuous on $\overline{D}$. Assume that $\operatorname{Re} f(z) = \operatorname{Re} g(z)$ for all $z \in \partial D$. Prove that $f - g$ is constant.
20. Let f be analytic on $D = D(0, 1)$. Prove that either f has a zero in D, or there is a sequence $\{z_n\}$ in D such that $|z_n| \to 1$ and $\{f(z_n)\}$ is bounded.
21. Let u be a nonnegative harmonic function on $\mathbb{C}$. Prove that f is constant.
22. Suppose f is analytic on $\Omega \supseteq \overline{D}(0, 1), f(0) = i$, and $|f(z)| > 1$ whenever $|z| = 1$. Prove that f has a zero in $D(0, 1)$.
23. Find the maximum value of $\operatorname{Re} z^3$ for z in the unit square $[0, 1] \times [0, 1]$.

24. Suppose that f is analytic on $D(0,1)$, with $f(0) = 0$. Define $f_n(z) = f(z^n)$ for $n = 1, 2, \ldots, z \in D(0,1)$. Prove that $\sum f_n$ is uniformly convergent on compact subsets of $D(0,1)$. (Use Schwarz's lemma.)

25. It follows from (2.4.12c) that if f is analytic on $D(0,1)$ and $f(z_n) \to 0$ for each sequence $\{z_n\}$ in $D(0,1)$ that converges to a point of $C(0,1)$, then $f \equiv 0$. Prove the following strengthened version for *bounded* f. Assume only that $f(z_n) \to 0$ for each sequence $\{z_n\}$ that converges to a point in some given arc $\{e^{it}, \alpha \leq t \leq \beta\}$ where $\alpha < \beta$, and deduce that $f \equiv 0$. [Hint: Assume without loss of generality that $\alpha = 0$. Then for sufficiently large n, the arcs $A_j = \{e^{it} : (j-1)\beta \leq t \leq j\beta\}, j = 1, 2, \ldots, n$ cover $C(0,1)$. Now consider $F(z) = f(z)f(e^{i\beta}z)f(e^{i2\beta}z) \cdots f(e^{in\beta}z)$.]

26. (a) Let Ω be a bounded open set and let $\{f_n\}$ be a sequence of functions that are analytic on Ω and continuous on the closure $\overline{\Omega}$. Suppose that $\{f_n\}$ is uniformly Cauchy on the boundary of Ω. Prove that $\{f_n\}$ converges uniformly on $\overline{\Omega}$. If f is the limit function, what are some properties of f?
    (b) What complex-valued functions on the unit circle $C(0,1)$ can be uniformly approximated by polynomials in z?