CONTRACTION PRINCIPLE

Proposition 0.1. *Suppose that $\{X^\varepsilon;\, \varepsilon > 0\}$ is a collection of random variables defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and taking values in some Polish space E. Suppose furthermore that $\{X^\varepsilon;\, \varepsilon > 0\}$ has a large deviations principle with rate function I. Suppose that E' is a second Polish space and suppose that ψ is a continuous map from E to E'. Define the E'-valued random variables $Y^\varepsilon \overset{def}{=} \psi(X^\varepsilon)$ for all $\varepsilon > 0$. Then $\{Y^\varepsilon;\, \varepsilon > 0\}$ has a large deviations principle with rate function*

$$I'(y) \overset{def}{=} \inf\{I(x) : \psi(x) = y\}.$$

Proof. We break the calculation up into three steps.
Upper Bound. Fix a closed subset F of E'. Then

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{Y^\varepsilon \in F\} = \varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X^\varepsilon \in \psi^{-1}(F)\} \le -\inf\left\{I(x) : x \in \psi^{-1}(F)\right\} = -\inf\left\{I'(y) : y \in F\right\}.$$

Lower Bound. Fix an open subset G of E'. Then

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{Y^\varepsilon \in G\} = \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X^\varepsilon \in \psi^{-1}(G)\} \ge -\inf\left\{I(x) : x \in \psi^{-1}(G)\right\} = -\inf\left\{I'(y) : y \in G\right\}.$$

Compactness. Since ψ is continuous, it suffices to show that for any $s \ge 0$,

$$\Phi'(s) \overset{\text{def}}{=} \{y \in E' : I'(y) \le s\} = \psi(\Phi(s)).$$

First, we show that $\psi(\Phi(s)) \subset \Phi'(s)$. If $x \in \Phi(s)$, then $I(x) \le s$, so if we define $y \overset{\text{def}}{=} \psi(x)$, we see that $x \in \{x' \in E : \psi(x') = y\}$, so $I'(y) \le I(x) \le s$, so indeed $\psi(\Phi(s)) \subset \Phi'(s)$.

Next, we show that $\Phi'(s) \subset \psi(\Phi(s))$. Fix $y \in \Phi'(s)$. Thus, for each $n \in \mathbb{N}$, there is an $x_n \in E$, such that $\psi(x_n) = y$ and

$$I(x_n) \le I'(y) + \frac{1}{n} \le s + \frac{1}{n}.$$

Since $\Phi(s+1)$ is compact, there is a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with limit point x^*. By continuity, $\psi(x^*) = \lim_k \psi(x_{n_k}) = y$. Fixing any $\delta > 0$, we see that for k large enough that $n_k \ge 1/\delta$, $x_{n_k} \in \Phi(s + 1/n_k) \subset \Phi(s+\delta)$. Since $\Phi(s+\delta)$ is closed, we thus have that $x^* \in \Phi(s+\delta)$; i.e., $I(x^*) \le s + \delta$. Thus in fact, $I(x^*) \le s$. Hence $\psi(x^*) = y$ and $x^* \in \Phi(s)$, so $y \in \psi(\Phi(s))$. □