ENTROPY

Definition 0.1. Fix a Polish space X. For any μ and ν in $\mathscr{P}(X)$, define

$$H(\nu|\mu) \overset{\text{def}}{=} \begin{cases} \int_{z\in X} \ln \frac{d\nu}{d\mu}(z)\nu(dz) & \text{if } \nu \ll \mu \\ \infty & \text{else,} \end{cases}$$

where the integral is interpreted in the sense of Lebesgue. Some introductory comments are in order.

Lemma 0.2. *For any Polish space X and any ν and μ in $\mathscr{P}(X)$, $H(\nu|\mu)$ is well-defined and nonnegative. We have that $H(\nu|\mu)=0$ if and only if $\nu=\mu$.*

Proof. Define $f(x) \overset{\text{def}}{=} x\ln x$ for all $x>0$, and define $f(0)=0$; it is easy to see that f is convex and continuous on $[0,\infty)$.

We first claim that when $\nu \ll \mu$, $H(\nu|\mu)$ is well-defined. To do this, we will show that

$$\int_{z\in A} \left\{-\ln \frac{d\nu}{d\mu}(z)\right\}\nu(dz) < \infty.$$

where

$$A \overset{\text{def}}{=} \left\{z\in X : 0 \le \frac{d\nu}{d\mu}(z) \le 1\right\}.$$

If $\nu(A)=0$, then this integral is zero (by standard construction of Lebesgue integrals as limits of integrals of increasing simple functions). If $\nu(A)>0$, then $\mu(A)>0$. The convexity of f implies that $-f$ is concave, so

$$\begin{aligned}\int_{z\in A} \left\{-\ln \frac{d\nu}{d\mu}(z)\right\}\nu(dz) &= \int_{z\in A}\left\{-f\left(\frac{d\nu}{d\mu}(z)\right)\right\}\mu(dz) \\ &\le -f\left(\frac{\int_{z\in A}\frac{d\nu}{d\mu}(z)\mu(dz)}{\mu(A)}\right)\mu(A) \le -f\left(\frac{\nu(A)}{\mu(A)}\right)\mu(A) < \infty,\end{aligned}$$

proving that $H(\nu|\mu)$ is well-defined. To show that it is positive, we again use the convexity of f to see that

$$\int_{z\in X} \ln\frac{d\nu}{d\mu}(z)\nu(dz) = \int_{z\in X} f\left(\frac{d\mu}{d\nu}(z)\right)\mu(dz) \ge f\left(\int_{z\in X}\frac{d\mu}{d\nu}(z)\mu(dz)\right) = f(1) = 0.$$

Since f is strictly convex, equality holds if and only if $\frac{d\nu}{d\mu}$ is constant, in which case $\frac{d\nu}{d\mu}\equiv 1$, and thus $H(\nu|\mu)=0$. ☐

Theorem 0.3 (Entropy Duality)**.** *Fix a Polish space X. For any ν and μ in $\mathscr{P}(X)$, we have that*

$$H(\nu|\mu) = \sup_{\phi\in B(X)}\left\{\int_{z\in X}\phi(z)\nu(dz) - \log\int_{z\in X} e^{\phi(z)}\mu(dz)\right\} \tag{1}$$

and for any $\phi\in B(X)$, we have that

$$\log\int_{z\in X} e^{\phi(z)}\mu(dz) = \sup_{\nu\in\mathscr{P}(X)}\left\{\int_{z\in X}\phi(z)\nu(dz) - H(\nu|\mu)\right\}. \tag{2}$$

Proof. First note that if $\nu\in\mathscr{P}(X)$ is such that $H(\nu|\mu)<\infty$ and $\phi\in B(X)$, then by Jensen's inequality

$$\begin{aligned}\int_{z\in X}\phi(z)\nu(dz) - H(\nu|\mu) &= \int_{z\in X}\left\{\phi(z) - \ln\frac{d\nu}{d\mu}(z)\right\}\nu(dz) \\ &= \int_{\substack{z\in X \\ \frac{d\nu}{d\mu}(z)>0}} \ln\frac{e^{\phi(z)}}{\frac{d\nu}{d\mu}(z)}\nu(dz) \le \ln\int_{z\in X} e^{\phi(z)}d\mu(z)\end{aligned}$$

(note that $\nu\left\{z\in X : \frac{d\nu}{d\mu}(z)>0\right\}=1$). Thus the left-hand sides of (1) and (2) are greater than or equal to the right-hand sides.

Let's next show that the left-hand side of (1) is less than the right-hand side. Define $\nu\in\mathscr{P}(X)$ via

$$\nu(A)\overset{\text{def}}{=}\frac{\int_{z\in A}e^{f(z)}\mu(dz)}{\int_{z\in X}e^{f(z)}\mu(dz)}. \qquad A\in\mathscr{B}(X)$$

Then

$$\int_{z\in X}\phi(x)\nu(dx)-\ln\int_{z\in X}e^{\phi(z)}\mu(dz)=H(\nu|\mu).$$

Thus the left-hand side of (2) is less than or equal to the right-hand side.

The final step is to show that the left-hand side of (1) is less than or equal to the right-hand side. To do so, first assume that $\nu\ll\mu$. We will need to approximate. Define

$$\phi_n(x)\overset{\text{def}}{=}\begin{cases}\ln\frac{d\nu}{d\mu}(x) & \text{if } |\ln\frac{d\nu}{d\mu}(x)|<N\\ N & \text{if } \ln\frac{d\nu}{d\mu}(x)>N\\ -N & \text{if } \ln\frac{d\nu}{d\mu}(x)<-N.\end{cases}$$

Then we have that

$$\lim_n\int_{z\in X}\phi_n(z)\nu(dz)=H(\nu|\mu).$$

We also note that

$$0\le e^{\phi_n(x)}\le 1+\frac{d\nu}{d\mu}(x);$$

thus by dominated convergence,

$$\lim_n\ln\int_{z\in E}e^{\phi_n(z)}\mu(dz)=0.$$

Thus

$$H(\nu|\mu)=\lim_n\left\{\int_{z\in X}\phi_n(z)\nu(dz)-\ln\int_{z\in E}e^{\phi_n(z)}\mu(dz)\right\},$$

which proves that the left-hand side of (2) is less than or equal to the right-hand side in this case (when $\nu\ll\mu$). Finally, assume that $\nu\not\ll\mu$. Then $H(\nu|\mu)=\infty$ and there is an $A\in\mathscr{B}(X)$ such that $\nu(A)>0$. Define now $\phi_n\overset{\text{def}}{=}n\chi_A$ for all n. Then

$$\varlimsup_n\left\{\int_{z\in X}\phi_n(z)\nu(dz)-\ln\int_{z\in E}e^{\phi_n(z)}\mu(dz)\right\}=\varlimsup_n n\nu(A)=\infty$$

This finishes the proof. □