On Conservative Systems Perturbed by Noise

Richard B. Sowers
Department of Mathematics
University of Illinois
Urbana-Champaign
http://www.math.uiuc.edu/~r-sowers

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$$\underbrace{\textbf{Differential Equations}} + \underbrace{\textbf{Small Noise}}$$

classical theory | thermal fluctuations

complex behaviors | subscale uncertainty

fundamental modelling tool

Q: What happens if we add small noise? $\varepsilon\Delta$

- Look primarily at *conserved quantities* (reduction problems)
- Dynamics can by *complex* (bifurcations)

We are interested in the competition between complicated dynamics and the regularizing effect of small noise.

Outline

- Reduction Results
  - Warmup on Cylinder/Single-Well Potential (review and notation)
    - Khasminskii (1960's)
    - no critical points or elliptic critical points
  - Double-Well Potential
    - Neishtadt, Freidlin-Wentzell, Freidlin-Weber (1990's)
    - hyperbolic critical points
  - Whiskered Sphere
    - S (2002)
    - region of critical points
- Reduction Techniques
  - Singular Perturbations analysis (S, new)
- Related/Future Problems

Noisy Simple Harmonic Oscillator (Single-Well Potential)

Let's start with the equation of a randomly-perturbed **1-dimensional** particle in a force field

$$dx_t^\varepsilon = p_t^\varepsilon dt$$
$$dp_t^\varepsilon = -U'(x_t^\varepsilon)dt + \varepsilon dW_t$$

Slowly-varying quantity is

$$\boxed{H(x,p) \stackrel{\text{def}}{=} \frac{p^2}{2} + U(x)}$$

What are the dynamics of the conserved quantity

(energy) as $\varepsilon \to 0$?

Visible fluctuations of H occur on time scales of size ε^{-2}, define $Y_t^\varepsilon \stackrel{\text{def}}{=} (x^\varepsilon_{t/\varepsilon^2}, p^\varepsilon_{t/\varepsilon^2})^T$. Then

$$\boxed{dY_t^\varepsilon = \frac{1}{\varepsilon^2}\nabla^\perp H(Y_t^\varepsilon)dt + \varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t}$$

where $\nabla^\perp H = (\partial H/\partial p, -\partial H/\partial x)$.

Convert to angular coordinates; we get

$$d\theta_t^\varepsilon = \frac{1}{\varepsilon^2}\omega(h_t^\varepsilon)dt + \text{smaller terms}$$
$$dh_t^\varepsilon = b(\theta_t^\varepsilon, h_t^\varepsilon)dt + \sigma(\theta_t^\varepsilon, h_t^\varepsilon)dW_t$$

where

$$h_t^\varepsilon \overset{\text{def}}{=} H(Y_t^\varepsilon)$$

Then $h^\varepsilon \overset{\varepsilon\searrow 0}{\Longrightarrow} h^0$ (weak convergence of processes) where

$$dh_t^0 = \bar{b}(h_t^0)dt + \bar{\sigma}^2(h_t^0)dW_t$$

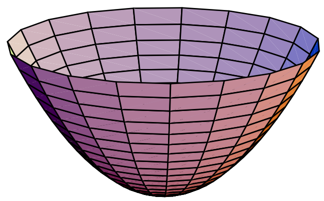
with

$$\bar{b}(h) \overset{\text{def}}{=} \int_0^1 b(h,\theta)d\theta \qquad \text{and} \qquad \bar{\sigma}^2(h) \overset{\text{def}}{=} \int_0^1 \sigma^2(h,\theta)d\theta.$$

The generator of h^0 is

$$\bar{\mathscr{L}} \overset{\text{def}}{=} \bar{b}(h)\frac{d}{dh} + \frac{1}{2}\bar{\sigma}^2(h)\frac{d^2}{dh}$$

Picture is



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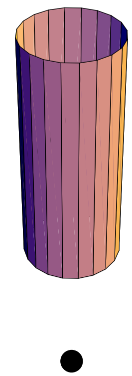


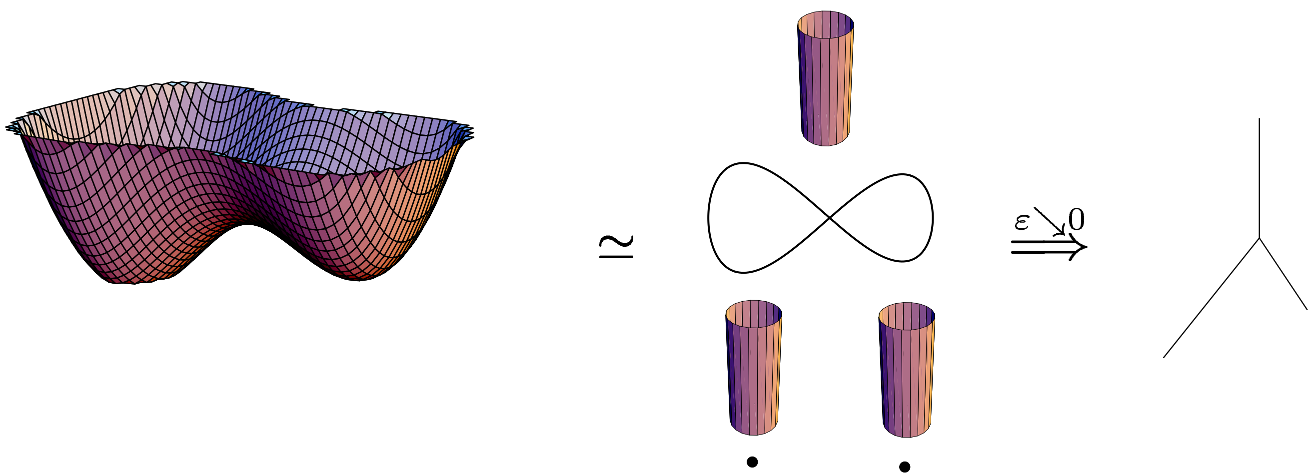
$\xrightarrow{\varepsilon \searrow 0}$

Double-Well Potential

Consider

$$dY_t^\varepsilon = \frac{1}{\varepsilon^2}\nabla^\perp H(Y_t^\varepsilon)dt + \varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t$$

where H is a double-well potential.



Here, to get a reduced model, keep track of

$$\xi_t^\varepsilon = \underbrace{(H(Y_t^\varepsilon), \mathsf{Leg}(Y_t^\varepsilon))}_{\mathbb{R}\times\{1,2,3\}}$$

Then

$$\xi^\varepsilon \overset{\varepsilon\searrow 0}{\Longrightarrow} \xi^0$$

where ξ^0 takes values on a “wye”-shaped graph Γ (with vertex $\star$) Generator of ξ^0 is

$$\hat{\mathscr{L}} f(h,i) = \bar{\mathscr{L}}_i f(h,i)$$

and $\hat{\mathscr{L}}$ has domain

$$\hat{\mathscr{D}}(\hat{\mathscr{L}}) = \Big\{ f \in C(\Gamma) \cap C^2(\Gamma \setminus \star) : \hat{\mathscr{L}} f \in C(\Gamma) \text{ and } \sum_{i=1}^{3} \mathrm{G}_i \frac{\partial f}{\partial h}(0,i) = 0 \Big\}$$

- G_i's are *glueing coefficients*
- G_i's define a *coin flip*

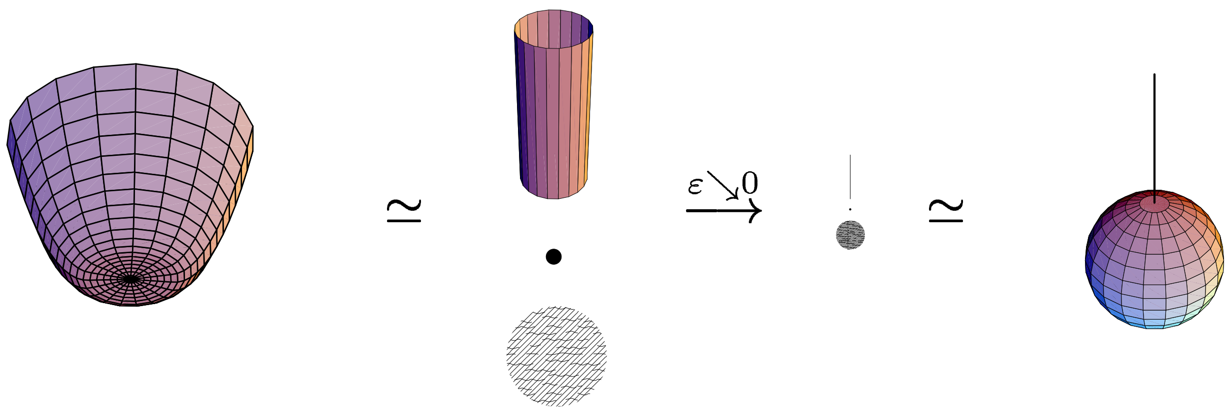
Whiskered Sphere (S, 2002)

$$dY_t^\varepsilon = \frac{1}{\varepsilon^2}\nabla^\perp H(Y_t^\varepsilon)dt + \varepsilon \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} dW_t$$

where

$$H(y) \overset{\text{def}}{=} (\max\{\|y\| - 1, 0\})^{p(>2)}$$

Here $\nabla^\perp H \equiv 0$ on $D^2 \overset{\text{def}}{=} \{y \in \mathbb{R}^2 : \|y\| \leq 1\}$.



Reduced process is on $\mathbb{R}_+ \cup D^2$ (dimensional discontinuity). Glueing conditions here are a coin flip and an entrance measure.

Moral: Critical points (Morse data) in conserved quantities lead to stratification of the reduced space (via *chain equivalence*). Extra glueing conditions must be identified at vertices (Evans-S).

Reduction Technique

Analysis for double-well problem (for simplicity)

- new natural interpretation and proof
- works when noise is degenerate (but not discussed here)

Generator of $\varepsilon > 0$ process is

$$\boxed{\boxed{\mathscr{L}^\varepsilon \overset{\text{def}}{=} \frac{1}{\varepsilon^2}\nabla^\perp H + \frac{1}{2}\Delta}}$$

Exit of a diffusion from a domain

$$\Uparrow$$

Harmonic Functions $\mathscr{L}^\varepsilon u^\varepsilon \approx 0$ and *boundary data*

Posit that

$$u^{\varepsilon}(x) = U\left(\Theta(x), \frac{H(x)}{\varepsilon^{\alpha}}\right)$$

- Θ is boundary coordinate along homoclinic orbit
- ε^{α} gives width of boundary layer

$$(\mathscr{L}^{\varepsilon} u^{\varepsilon})(x) = \frac{1}{\varepsilon^2}\frac{\partial U}{\partial \theta}(\nabla^{\perp} H, \nabla \Theta) + \frac{1}{2\varepsilon^{\alpha}}\frac{\partial^2 U}{\partial h^2}\|\nabla H\|^2 + \text{smaller terms}$$

- $\alpha = 1$ (so boundary layer is $H = O(\varepsilon)$)
- $(\nabla^{\perp} H, \nabla \Theta) = \|\nabla H\|^2$
  - "Khasminskii coordinates"
  - equal increments of Θ correspond to equal increments of transversal diffusion across $H^{-1}(0)$.
- U satisfies Heat Equation

$$\frac{\partial U}{\partial \theta} + \frac{1}{2}\frac{\partial^2 U}{\partial h^2} \equiv 0$$

Glueing Coefficients

$\Updownarrow$

Likelihood of going into region R_i

$\Updownarrow$

Relative length of ∂R_i in Θ-coordinates (EXPLICIT FORMULAE)

Idea of Rigorous Proof

Ingredients:

- Original generator $\mathscr{L}^\varepsilon$ with domain $C^2(\mathbb{R}^2)$
- map π from $\mathbb{R}^2$ to graph Γ
- test function f in $\hat{\mathscr{D}}(\hat{\mathscr{L}}) \subset B(\Gamma)$

Note:

- $f \circ \pi$ is **not** in $C^2(\mathbb{R}^2)$ (it is in $C(\mathbb{R}^2) \setminus C^1(\mathbb{R}^2)$).

$$f(\pi(x)) \approx f(\star) + \dot{f}_i(0)H(x) \qquad \text{near } \partial R_i$$

Make a **perturbed test function** (Kushner, Evans, Kurtz) such that

- $f \circ \pi + \Phi^\varepsilon \in C^1(\mathbb{R}^2)$ (hidden boundary conditions)
- $\mathscr{L}^\varepsilon \Phi^\varepsilon \approx 0$
- $\Phi^\varepsilon \approx 0$

Claim: this is doable if and only if $\sum_i \mathrm{G}_i \dot{f}_i(0) = 0$.

$$\Phi^{\varepsilon}(x) = \varepsilon U_i \left(\Theta(x), \frac{H(x)}{\varepsilon} \right) \qquad \text{on } R_i$$

We want

$$\frac{\partial U}{\partial \theta} + \frac{1}{2} \frac{\partial^2 U}{\partial h^2} \equiv 0$$

U_i's are appropriate θ-periodic

U_i's are bounded

$$U_1(\theta, 0) = \begin{cases} U_2(\theta, 0) & \text{on } \partial R_1 \cap \partial R_2 \\ U_3(\theta, 0) & \text{on } \partial R_1 \cap \partial R_3 \end{cases} \quad \text{(Dirichlet data matches)}$$

$$\frac{\partial U_1}{\partial h}(\theta, 0) + \dot{f}_1(0) = \begin{cases} \frac{\partial U_2}{\partial h}(\theta, 0) + \dot{f}_2(0) & \text{on } \partial R_1 \cap \partial R_2 \\ \frac{\partial U_3}{\partial h}(\theta, 0) + \dot{f}_3(0) & \text{on } \partial R_1 \cap \partial R_3 \end{cases} \quad \text{(Neumann data matches)}$$

Leads to Hilbert-space equation using Neumann-Dirichlet operators. Solvable iff $\dot{f}_i(0)$'s satisfy glueing conditions (Fredholm-alternative).

Related/Future Problems

- Engineering-type calculations and time-periodic terms (Namachchivaya and S).
- Stability (S, Baxendale-Goukasian).

$$\|DY_t^\varepsilon\| \asymp \exp\left[\varepsilon^{-4/3}\int_0^t \lambda(H(Y_s^\varepsilon))ds\right]$$

- Delay at vertex (in progress).
- Effect of noise upon other types of bifurcations (student).
- Discontinuous systems (Dupuis).
- Very complicated dynamical systems.