RIGOROUS STOCHASTIC AVERAGING ON A CYLINDER

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ABSTRACT. We demonstrate how to use the martingale problem to rigorously average an SDE on a cylinder with fast angular drift and slow axial diffusion.

1. INTRODUCTION

Let W be a Brownian motion on some (original) probability space $(\Omega^\circ, \mathscr{F}^\circ, \mathbb{P}^\circ)$. Consider the $\mathbb{R}^2$-valued stochastic differential equation

$$\begin{aligned} dX_t^\varepsilon &= \frac{1}{\varepsilon^2}dt \\ dY_t^\varepsilon &= \sigma(X_t^\varepsilon)dW_t \\ X_0^\varepsilon &= x_\circ \\ Y_0^\varepsilon &= y_\circ \end{aligned} \tag{1}$$

where $x_\circ$ and $y_\circ$ are some fixed elements of $\mathbb{R}$ and where σ is a bounded element of $C^\infty(\mathbb{R})$. We furthermore assume *periodicity*; that

$$\sigma(x) = \sigma(x+1)$$

for all x in $\mathbb{R}$. Note that we explicitly have that

$$Y_t^\varepsilon = y_\circ + \int_0^t \sigma(x_\circ + u/\varepsilon^2)dW_u$$

for all $t > 0$. See Figure 1. We are interested in the behavior of Y^ε as ε tends to zero. Specifically, we will prove

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Given at the GASP seminar on June 19th, 2001. This is not original research, but rather a restatement of a classical result in the framework of the martingale problem. These notes are intended to introduce the advanced graduate student to the mechanics of the martingale problem.

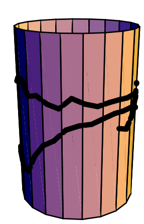

FIGURE 1. A Trajectory on the Cylinder

Theorem 1.1 (Main Theorem). *We have that Y^ε converges in distribution to Y^0, where Y^0 satisfies the stochastic differential equation*

$$dY^0_t = \bar\sigma(Y^0_t)dW_t$$
$$Y^0_0 = y_\circ$$

where

$$\bar\sigma \overset{def}{=} \left\{\int_0^1 \sigma^2(x)dx\right\}^{1/2.}$$

We note that Y^0 can be explicity written as

$$Y^0 = y_\circ + \bar\sigma W_t$$

for all $t > 0$. We will prove the above theorem via the *martingale problem* (see [1]). This result is due to Khasminskii [2].

2. Martingale Problem

First, let's write the dynamics of $Z^\varepsilon_t \overset{\text{def}}{=} (X^\varepsilon_t, Y^\varepsilon_t)$ via the martingale problem. Ito's formula tells us that for any $f \in C^2_0(\mathbb{R}^2)$, we have that

$$f(Z^\varepsilon_t) = f(Z^\varepsilon_0) + \int_0^t \mathscr{L}^\varepsilon f(Z^\varepsilon_u)du + M^{\varepsilon,f}_t$$

where

$$(\mathscr{L}^\varepsilon f)(x,y) \overset{\text{def}}{=} \frac{1}{\varepsilon^2}\frac{\partial f}{\partial x}(x,y) + \frac{\sigma^2(x)}{2}\frac{\partial^2 f}{\partial y^2}(x,y) \qquad (x,y) \in \mathbb{R}^2$$
$$M^{\varepsilon,f}_t \overset{\text{def}}{=} \int_0^t \sigma(X^\varepsilon_u)\frac{\partial f}{\partial y}(X^\varepsilon_u)dW_u. \qquad t > 0$$

In particular, the martingale claim follows from standard results of stochastic integration; we have that

$$\mathbb{E}^\circ\left[M^{\varepsilon,f}_t \middle| Z^\varepsilon_r;\ 0 \le r \le s\right] = M^{\varepsilon,f}_s$$

for all $0 \le s \le t$ (we note that the filtration generated by Z^ε is a sub-filtration of that generated by W). Note that this is equivalent to the fact that

$$\mathbb{E}^\circ\left[M^{\varepsilon,f}_t - M^{\varepsilon,f}_s \middle| Z^\varepsilon_r;\ 0 \le r \le s\right] = 0.$$

We note now that

$$M^{\varepsilon,f}_t - M^{\varepsilon,f}_s = f(Z^\varepsilon_t) - f(Z^\varepsilon_s) - \int_s^t \mathscr{L}^\varepsilon f(Z^\varepsilon_u)du.$$

In particular, this means that for any $0 \le r_1 < r_2 \cdots < r_n \le s \le t$ and $\{\varphi_j;\ j = 1,2\dots n\} \subset C_b(\mathbb{R}^2)$, then

$$\mathbb{E}^\circ\left[\left\{f(Z^\varepsilon_t) - f(Z^\varepsilon_s) - \int_s^t (\mathscr{L}^\varepsilon f)(Z^\varepsilon_u)du\right\}\prod_{j=1}^n \varphi_j(Z^\varepsilon_{r_j})\right] = 0. \tag{2}$$

We shall use this as our starting point. For future reference, let's explicitly write down the generator of (1).

Definition 2.1. The generator of (1) is $\mathscr{L}^\varepsilon$, with domain $\mathscr{D}(\mathscr{L}^\varepsilon) \supset C^2_0(\mathbb{R}^2)$.

3. Goal

We now define our goal. For each $\varepsilon > 0$, define

$$\bar{\mathbb{P}}^\varepsilon(A) \stackrel{\text{def}}{=} \mathbb{P}^\circ\{Y^\varepsilon \in A\}$$

for all $A \in \mathscr{B}(C([0,\infty);\mathbb{R}))$; thus $\bar{\mathbb{P}}^\varepsilon \in \mathscr{P}(C([0,\infty);\mathbb{R})$ for all $\varepsilon > 0$. We want to assert the existence of and identify the limit

$$\bar{\mathbb{P}}^0 \stackrel{\text{def}}{=} \lim_{\varepsilon\searrow 0} \bar{\mathbb{P}}^\varepsilon, \tag{3}$$

this limit being in the Prohorov topology on $\mathscr{P}(C([0,\infty);\mathbb{R}))$. Let's make a canonical setup on $C([0,\infty);\mathbb{R})$. Define the event space $\Omega \stackrel{\text{def}}{=} C([0,\infty);\mathbb{R})$. Define the coordinate functions $\mathfrak{y}_t(\omega) \stackrel{\text{def}}{=} \omega(t)$ for all $t \geq 0$ and all $\omega \in \Omega$. For each $t \geq 0$, define $\mathscr{F}_t \stackrel{\text{def}}{=} \sigma\{\mathfrak{y}_s;\ 0 \leq s \leq t\}$ and define a sigma-algebra on Ω by $\mathscr{F} \stackrel{\text{def}}{=} \bigvee_{t\geq 0} \mathscr{F}_t$. Define also the generator

$$\bar{\mathscr{L}} f(y) \stackrel{\text{def}}{=} \frac{\bar{\sigma}^2}{2}\frac{\partial^2 f}{\partial y^2}(y)$$

for all $f \in C_0^2(\mathbb{R})$ and all $y \in \mathbb{R}$. We want to show several things. Firstly, we want to show that the $\bar{\mathbb{P}}^\varepsilon$'s are tight, so that at least one limit point in (3) exists. Secondly, we want to show that $0 \leq r_1 < r_2 \dots < r_n \leq s \leq t$ and $\{\varphi_j;\ j = 1, 2\dots n\} \subset C_b(\mathbb{R})$ and $f \in C_0^2(\mathbb{R})$,

$$\bar{\mathbb{E}}^0\left[\left\{f(\mathfrak{y}_t) - f(\mathfrak{y}_s) - \int_s^t (\bar{\mathscr{L}} f)(\mathfrak{y}_u)du\right\}\prod_{j=1}^n \varphi_j(\mathfrak{y}_{r_j})\right] = 0.$$

This means that $\bar{\mathbb{P}}^0$ satisfies the martingale problem with $\bar{\mathscr{L}}$ whose domain is $\mathscr{D}(\bar{\mathscr{L}}) \supset C_0^2(\mathbb{R})$. Thirdly, we want to show that there is only one solution of this martingale problem, so that we have uniquely characterized the limit and the limit is Markovian.

4. Tightness

First, we want to show

Proposition 4.1 (Tightness). *For each $\delta > 0$ and $T > 0$,*

$$\varlimsup_{\eta\searrow 0} \sup_{0<\varepsilon<1} \bar{\mathbb{P}}^\varepsilon\left\{\sup_{\substack{0\leq s<t\leq T\\ |s-t|\leq\eta}} |\mathfrak{y}_t - \mathfrak{y}_s| \geq \delta\right\} = 0.$$

Thus, the $\bar{\mathbb{P}}^\varepsilon$'s are tight in the Prohorov topology on $\mathscr{P}(C([0,\infty);\mathbb{R}))$.

Proof. We have that

$$\bar{\mathbb{P}}^\varepsilon\left\{\sup_{\substack{0\leq s<t\leq T\\ |s-t|\leq\eta}} |Y_t - Y_s| \geq \delta\right\} = \mathbb{P}^\circ\left\{\sup_{\substack{0\leq s<t\leq T\\ |s-t|\leq\eta}} |M_t^\varepsilon - M_s^\varepsilon| \geq \delta\right\}$$

where

$$M_t^\varepsilon = \int_0^t \sigma(X_u^\varepsilon)dW_u$$

for all $t > 0$. We use Levy's theorem. Fix $\varepsilon > 0$; then there is a Brownian motion B such that

$$M_t^\varepsilon = B_{\langle M^\varepsilon \rangle_t}$$

for all $t > 0$, where

$$\langle M^\varepsilon \rangle_t = \int_0^t \sigma^2(X_u^\varepsilon) du \qquad t > 0$$

is the quadratic variation of M^ε. Then for any $0 \le s < t \le T$ such that $|s-t| \le \eta$,

$$|\langle M^\varepsilon \rangle_t - \langle M^\varepsilon \rangle_s| = \int_s^t \sigma^2(X_u^\varepsilon) du \le \|\sigma\| |t-s|$$

where $\|\sigma\|$ denotes the supremum norm of σ. Thus

$$|M_t^\varepsilon - M_s^\varepsilon| \le \sup_{\substack{0 \le s' < t' \le \|\sigma\| T \\ |t'-s'| \le \|\sigma\| \eta}} |B_{t'} - B_{s'}|.$$

Hence

$$\varlimsup_{\eta \searrow 0} \mathbb{P}^\circ \left\{ \sup_{\substack{0 \le s < t \le T \\ |s-t| \le \eta}} |M_t^\varepsilon - M_s^\varepsilon| \ge \delta \right\} \le \varlimsup_{\eta \searrow 0} \mathbb{P}^\circ \left\{ \sup_{\substack{0 \le s' < t' \le \|\sigma\| T \\ |t'-s'| \le \|\sigma\| \eta}} |B_{t'} - B_{s'}| > \eta \right\} = 0.$$

□

5. Convergence

We next want to prove

Proposition 5.1 (Limiting Martingale Problem). *Let $\bar{\mathbb{P}}^* \in \mathscr{P}(C([0,\infty);\mathbb{R}))$ be a cluster point of the $\bar{\mathbb{P}}^\varepsilon$'s. Let $\bar{\mathbb{E}}^*$ be the associated expectation operator. For any $0 \le r_1 < r_2 \cdots < r_n \le s \le t$ and $\{\varphi_j;\ j = 1, 2 \dots n\} \subset C_b(\mathbb{R})$ and any $f \in C_0^2(\mathbb{R})$,*

$$\bar{\mathbb{E}}^* \left[\left\{ f(\mathfrak{y}_t) - f(\mathfrak{y}_s) - \int_s^t (\bar{\mathscr{L}} f)(\mathfrak{y}_u) du \right\} \prod_{j=1}^n \varphi_j(\mathfrak{y}_{r_j}) \right] = 0. \tag{4}$$

Furthermore, $\bar{\mathbb{P}}^*\{\mathfrak{y}_0 = y_\circ\} = 1$. In other words, any cluster point of the $\bar{\mathbb{P}}^\varepsilon$'s satisfies the martingale problem associated with $\bar{\mathscr{L}}$ and with initial condition $\delta_{y_\circ}$.

We will first prove the following result.

Lemma 5.2. *The equality* (4) *holds if $f \in C_0^4(\mathbb{R})$.*

In other words, we want to show that for any r_j's, s, t, φ_j's, and f (either in $C_0^2(\mathbb{R})$ or $C_0^4(\mathbb{R})$),

$$\lim_{\varepsilon \searrow 0} \mathbb{E}^\circ \left[\left\{ f(Y_t^\varepsilon) - f(Y_s^\varepsilon) - \int_s^t (\bar{\mathscr{L}} f)(Y_u^\varepsilon) du \right\} \prod_{j=1}^n \varphi_j(Y_{r_j}^\varepsilon) \right] = 0.$$

The central calculation will be

Lemma 5.3 (Averaging). *Fix $\phi \in C_b^2(\mathbb{R}^2)$ which is 1-periodic in the first argument. Define*

$$(\mathbf{A}\phi)(y) \stackrel{def}{=} \int_0^1 \phi(x,y)dx$$

for all $y \in \mathbb{R}$. Then

$$\lim_{\varepsilon \searrow 0} \mathbb{E}^\circ \left[\left| \int_0^t \{\phi(X_u^\varepsilon, Y_u^\varepsilon) - (\mathbf{A}\phi)(Y_u^\varepsilon)\} \, du \right| \right] = 0.$$

Proof. Define

$$\Phi(x,y) \stackrel{\text{def}}{=} \int_0^x \{\phi(z,y) - (\mathbf{A}\phi)(y)\} \, dz$$

for all x and y in $\mathbb{R}^2$. By our definition of $\mathbf{A}\phi$, we have that Φ is bounded and 1-periodic in the first argument. We also can easily compute that

$$(\mathscr{L}^\varepsilon \Phi)(x,y) = \frac{1}{\varepsilon^2} \{\phi(x,y) - (\mathbf{A}\phi)(y)\} + \frac{\sigma^2(x)}{2} \frac{\partial^2 \Phi}{\partial y^2}(x,y)$$

Applying the martingale problem (2) to $\varepsilon^2 \Phi$, we have that

$$\varepsilon^2 \Phi(X_t^\varepsilon, Y_t^\varepsilon) - \varepsilon^2 \Phi(x_\circ, y_\circ) + \int_0^t \{\phi(X_u^\varepsilon, Y_u^\varepsilon) - (\mathbf{A}\phi)(Y_u^\varepsilon)\} \, du \\ + \frac{\varepsilon^2}{2} \int_0^t \sigma^2(X_u^\varepsilon) \frac{\partial^2 \Phi}{\partial y^2}(X_u^\varepsilon, Y_u^\varepsilon) du + \varepsilon^2 M_t$$

for all $t > 0$, where M is a martingale with quadratic variation; i.e.,

$$\int_0^t \{\phi(X_u^\varepsilon, Y_u^\varepsilon) - (\mathbf{A}\phi)(Y_u^\varepsilon)\} \, du = \varepsilon^2 \{\Phi(X_t^\varepsilon, Y_t^\varepsilon) - \Phi(x_\circ, y_\circ)\} \\ - \frac{\varepsilon^2}{2} \int_0^t \sigma^2(X_u^\varepsilon) \frac{\partial^2 \Phi}{\partial y^2}(X_u^\varepsilon, Y_u^\varepsilon) du - \varepsilon^2 M_t.$$

Simple calculations (involving Burkholder's inequality) give the desired result. □

We now give

Proof of Lemma 5.2. The martingale problem (2) tells us that

$$\mathbb{E}^\circ \left[\left\{ f(Y_t^\varepsilon) - f(Y_s^\varepsilon) - \int_s^t \frac{1}{2} \sigma^2(X_u^\varepsilon) \frac{\partial^2 f}{\partial y^2}(Y_u^\varepsilon) du \right\} \prod_{j=1}^n \varphi_j(Y_{r_j}^\varepsilon) \right] = 0.$$

for all $\varepsilon > 0$. It thus suffices to show that for any $t > 0$,

$$\lim_{\varepsilon \searrow 0} \mathbb{E}^\circ \left[\left| \int_0^t \left\{ \frac{1}{2} \sigma^2(X_u^\varepsilon) \frac{\partial^2 f}{\partial y^2}(Y_u^\varepsilon) - (\bar{\mathscr{L}})(Y_s^\varepsilon) \right\} du \right| \right] = 0.$$

This follows from Lemma 5.3 with

$$\phi(x,y) \stackrel{\text{def}}{=} \frac{\sigma^2(x)}{2} \frac{\partial^2 f}{\partial y^2}(y)$$

for all x and y in $\mathbb{R}$. Since f is assumed to be in $C_0^4(\mathbb{R})$, $\phi \in C_0^2(\mathbb{R})$. □

We now can fully prove Proposition 5.1.

Proof of Proposition 5.1. First, let's show that (4) holds for $f \in C_0^2(\mathbb{R})$. We will deduce this from the corresponding statement for $f \in C_0^2(\mathbb{R})$ by mollification. Let $\eta \in C_c^\infty(\mathbb{R};\mathbb{R}_+)$ be even and such that $\int_{\mathbb{R}} \eta(z)dz = 1$. Define

$$f_\delta(y) \overset{\text{def}}{=} \frac{1}{\delta}\int_{z\in\mathbb{R}} f(z)\eta\left(\frac{y-z}{\delta}\right)dz$$

for all $y \in \mathbb{R}$ and $\delta > 0$. Then $f_\delta \in C_0^4(\mathbb{R})$ for each $\delta > 0$, so

$$\bar{\mathbb{E}}^\varepsilon\left[\left\{f_\delta(\mathfrak{y}_t) - f_\delta(\mathfrak{y}_s) - \int_s^t (\bar{\mathscr{L}} f_\delta)(\mathfrak{y}_u)du\right\}\prod_{j=1}^n \varphi_j(\mathfrak{y}_{r_j})\right] = 0$$

for each $\delta > 0$. It is easy to see that $\|f_\delta - f\|_{C(\mathbb{R})}$ and $\|\bar{\mathscr{L}} f_\delta - \bar{\mathscr{L}} f\|_{C(\mathbb{R})}$ both tend to zero as δ tends to zero, so (4) holds for $f \in C_0^2(\mathbb{R})$. To finish, we only need to show that $\bar{\mathbb{P}}^*\{\mathfrak{y}_0 = y_\circ\} = 1$. Since $\{y_\circ\}$ is a closed set,

$$\bar{\mathbb{P}}^*\{\mathfrak{y}_0 = y_\circ\} \geq \overline{\lim_{\varepsilon\searrow 0}}\, \bar{\mathbb{P}}^\varepsilon\{\mathfrak{y}_0 = y_\circ\} = \overline{\lim_{\varepsilon\searrow 0}}\, \mathbb{P}^\circ\{Y_0^\varepsilon = y_\circ\} = 1.$$

□

6. Uniqueness

Proposition 6.1 (Uniqueness)**.** *There is only one solution of the martingale problem with generator $\bar{\mathscr{L}}$ with domain $\mathscr{D}(\bar{\mathscr{L}}) \supset C_0^2(\mathbb{R})$ and with initial condition $\delta_{y_\circ}$.*

This requires three results.

Lemma 6.2. *$C_0^2(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ (in the supremum norm).*

Proof. Classical □

Lemma 6.3. *The operator $\bar{\mathscr{L}}$ satisfies the positive maximum principle; for $f \in C_0(\mathbb{R})$ and $x^* \in \mathbb{R}$ such that $\sup_{x\in\mathbb{R}} f(x) = f(x^*) \geq 0$, we have that $(\bar{\mathscr{L}} f)(x^*) \leq 0$.*

Proof. Obvious □

Lemma 6.4. *Fix $\varphi \in C(\mathbb{R})$ and $\lambda > 0$. Then there is a $f \in \mathscr{D}(\bar{\mathscr{L}})$ such that*

$$\bar{\mathscr{L}} f - \lambda f = \varphi$$

Proof. Define

$$f(x) \overset{\text{def}}{=} -\int_0^\infty \int_{z\in\mathbb{R}} f(z)\frac{\exp\left[-\frac{(x-z)^2}{2t}\right]}{\sqrt{2\pi t}} e^{-\lambda t} dz\, dt$$

It is fairly easy to see that $f \in C_0^2(\mathbb{R})$ and that this f satisfies the desired PDE. □

References

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