Hamiltonian Systems with Noise: Glue, Spiders, and Lollipops

Richard B. Sowers
Department of Mathematics
University of Illinois
Urbana-Champaign
http://www.math.uiuc.edu/~ r-sowers

June 27, 2000

Outline

- Small Noise in Mechanics Problems
- Graph-valued problem
- Lollipop problem
- Other issues

Issues

- Convergence of Markov Processes/Semigroups
- Markov processes/PDE's on stratified spaces

Small Noise in Mechanics Problems

Let's start with the equation of a **1-dimensional** particle in a force field

$$\ddot{x}_t = F(x_t)$$

where $F \in C^\infty(\mathbb{R})$. This can be written as a **system of first-order ODE's**

$$\begin{aligned} \dot{x}_t^0 &= y_t^0 \\ \dot{y}_t^0 &= F(x_t^0). \end{aligned}$$

We can use the idea of **conservation of energy** to define

$$H(x, y) \stackrel{\text{def}}{=} \underbrace{U(x)}_{\text{potential energy}} + \underbrace{\frac{y^2}{2}}_{\text{kinetic energy}}$$

where $\dot{U}(x) = -F(x)$. Then

$$\begin{aligned} \dot{x}_t^0 &= \frac{\partial H}{\partial y}(x_t^0, y_t^0) \\ \dot{y}_t^0 &= -\frac{\partial H}{\partial x}(x_t^0, y_t^0) \end{aligned}$$

and

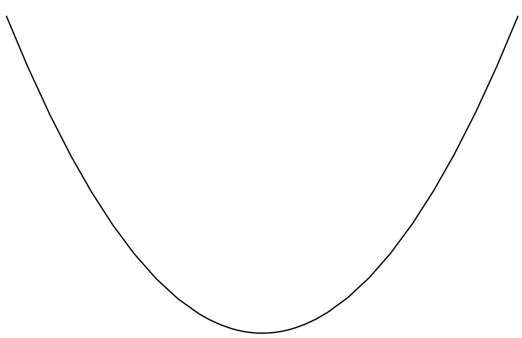
$$\frac{d}{dt}H(x_t^0, y_t^0) = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x}(x_t^0, y_t^0)$$
$$= \{H, H\}(x_t^0, y_t^0) = 0.$$

Thus the energy $H(x_t^0, y_t^0)$ is **conserved**.

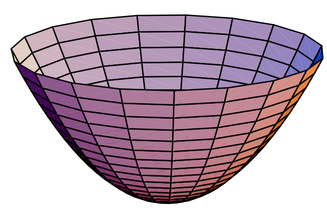
Let's assume for simplicity that the potential U has a single well, viz.

$$\boxed{U(x) = \frac{1}{2}x^2;}$$

thus the force is $F(x) = -x$.



Potential Energy

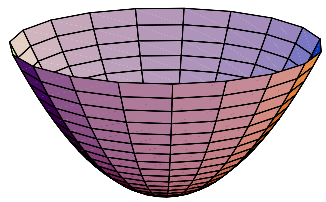


Total Energy

$$H(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$$

Single-Well Potential

We can understand the dynamics of (x_t^0, y_t^0) by looking at the **graph** of the energy H



$$H(x,y) = x^2/2 + y^2/2$$

The ODE for (x_t^0, y_t^0) simply moves around the level sets of H.

Let $x_t^0(x,y)$ and $y_t^0(x,y)$ be the solution of the particle dynamics with $x_0^0(x,y) = x$ and $y_0^0(x,y) = y$.

We note that if we set $h \overset{\text{def}}{=} H(x,y)$, then

$$\langle f, \mu_h \rangle \overset{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x_s^0(x,y), y_s^0(x,y))ds$$
$$= \frac{\int_{(x',y'):H(x',y')=h} f(x',y')\|\bar{\nabla} H(x',y')\|^{-1}\mathscr{H}(dx',dy')}{\int_{(x',y'):H(x',y')=h} \|\bar{\nabla} H(x',y')\|^{-1}\mathscr{H}(dx',dy')}.$$

In reality, the world is **noisy**; the acceleration is subject to random *jerk*. A more accurate model for the particle is

$$dx_t^\varepsilon = \frac{\partial H}{\partial y}(x_t^\varepsilon, y_t^\varepsilon)dt$$

$$dy_t^\varepsilon = -\frac{\partial H}{\partial x}(x_t^\varepsilon, y_t^\varepsilon)dt + \varepsilon dW_t,$$

where W is a Wiener process, and ε is a small parameter. Now energy is not conserved, but is **slowly-varying**; by Ito's formula,

$$dH(x_t^\varepsilon, y_t^\varepsilon) = \varepsilon \frac{\partial H}{\partial y}(x_t^\varepsilon, y_t^\varepsilon)dW_t + \underbrace{\varepsilon^2 \frac{1}{2}\frac{\partial^2 H}{\partial y^2}(x_t^\varepsilon, y_t^\varepsilon)dt}_{\text{Ito correction term}}$$

Thus, we will see a change in energy in time of order ε^{-2}; let's look at $(X_{t/\varepsilon^2}^\varepsilon, Y_{t/\varepsilon^2}^\varepsilon)$. This rescaling gives us a new SDE:

$$dX_t^\varepsilon = \frac{1}{\varepsilon^2}\frac{\partial H}{\partial y}(X_t^\varepsilon, Y_t^\varepsilon)dt$$

$$dY_t^\varepsilon = -\frac{1}{\varepsilon^2}\frac{\partial H}{\partial x}(X_t^\varepsilon, Y_t^\varepsilon)dt + dW_t.$$

Then

$$dH(X_t^\varepsilon, Y_t^\varepsilon) = \frac{\partial H}{\partial y}(X_t^\varepsilon, Y_t^\varepsilon)dW_t + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}(X_t^\varepsilon, Y_t^\varepsilon)dt.$$

Think of a **tornado**. If we put a tracer particle in a tornado, then we can't keep track of the angular position of the particle (it moves too fast), but we can see the vertical dynamics of the particle. We can try to find a **reduced model** for the dynamics of the vertical position. Somehow, we should **average** out the angular dynamics to get the vertical dynamics.

In the limit as ε tends to zero, we get that $H(X^\varepsilon) \to h$, where

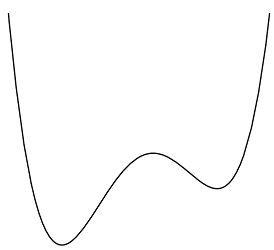
$$dh_t = b(h_t)dt + \sigma(h_t)dZ_t$$

for some Wiener process Z, where

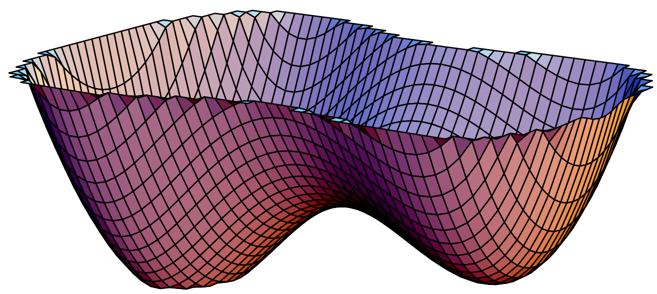
$$\sigma^2(h) \stackrel{\text{def}}{=} \left\langle \left(\frac{\partial H}{\partial y}\right)^2, \mu_h \right\rangle \quad \text{and} \quad b(h) \stackrel{\text{def}}{=} \left\langle \frac{1}{2}\frac{\partial^2 H}{\partial y^2}, \mu_h \right\rangle.$$

Graph-valued Problem

The foregoing is essentially **classical** and was understood in the 1960's (Khasminskii). A natural next problem, solved by Freidlin and Wentzell (1996) and Freidlin and Weber (1998) (see also Neishtadt (1991)) is that of a **double-well** potential.



Potential Energy

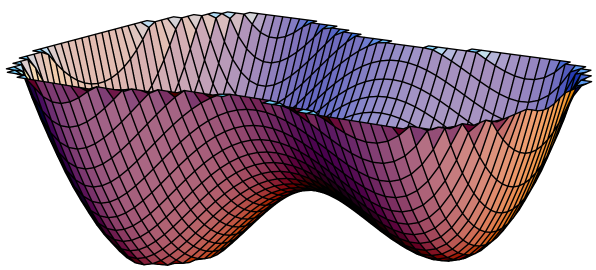


Total Energy Function

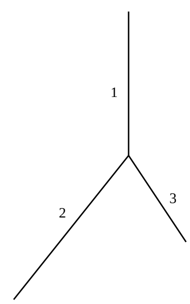
Double-Well Potential

If $h = H(x,y) > 0$, $\mu_{(x,y)}$ still depends only h, but if $h < 0$, it depends on h **and** which well we are interested in (i.e., whether $x < 0$ or $x > 0$).

Define an **equivalence relation** on $\mathbb{R}^2$; we say that $(x,y) \sim (x',y')$ if $\mu_{(x,y)} = \mu_{(x',y')}$. Then set $\Gamma \overset{\text{def}}{=} \mathbb{R}^2/\sim$.



becomes



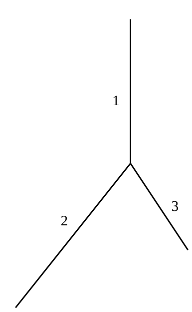
Γ

Geometry of Double-Well Case

Freidlin-Wentzell and Freidlin-Weber identified a limiting graph-valued process $\xi_t \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}[(X_t^\varepsilon, Y_t^\varepsilon)]$ (where $[(x,y)]$ is the equivalence class of (x,y) under $\sim$).

The limit ξ is given by the classical calculations as long as ξ_t does not touch the vertex. When it does, its behavior is governed by some **glueing conditions**, which are restrictions on the domain of the generator of the limiting process (more anon)

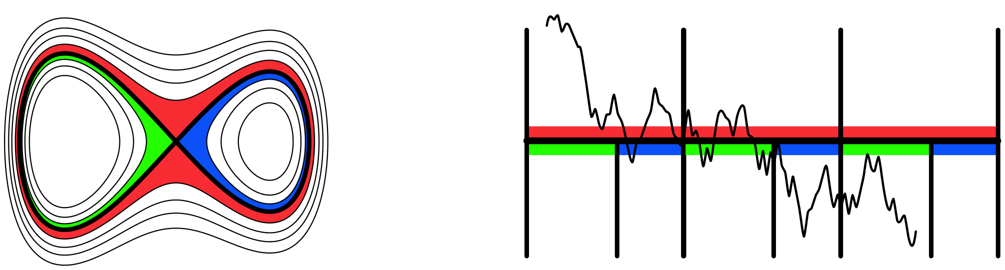
These glueing conditions are roughly and imprecisely as follows. If ξ starts at the vertex, one flips a three-sided coin (with statistics given by the glueing gonditions) to decided which leg to go to. Then ξ evolves according to the classical dynamics until it again hits the vertex, where the process is repeated again (with a new coin). This is similar to **skew Brownian motion**. It is also related to **spider martingales**.



One way (not the original way) to understand the glueing conditions is as follows. The process $(X^\varepsilon, Y^\varepsilon)$ moves between the different wells due to the diffusive kicks (i.e., the **martingale** part) of $H(X^\varepsilon, Y^\varepsilon)$. The bracket of this martingale is (dH, dH). Define an *angular* coordinate via PDE (due to Khasminskii)

$$(\nabla\Theta(x,y), \bar{\nabla} H)(x,y) = (dH, dH)(x,y).$$

Equal increments in Θ correspond to equal amounts of H-bracket. This allows us to change pictures and make a a boundary-layer expansion. The glueing conditions are exactly what is needed to solve a certain PDE on this space.



Khasminskii Coordinates for Graph-Valued Problem

New Problem

Let's make two changes to the system.

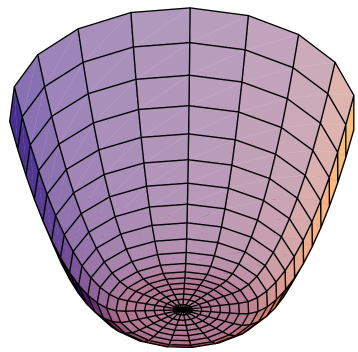
- In both the single-well and double-well problems, the critical points of H are **nondegenerate**; i.e., if $dH(x,y)=0$, then $D^2H(x,y)$ is full rank. We want to violently remove this assumption. We will assume that H has a **flat**.
- Let's assume that the noise is in both components (to avoid complications).

Our "canonical" Hamiltonian will be

$$H(z) \stackrel{\text{def}}{=} (\max\{\|z\|-1,0\})^n \qquad z\in\mathbb{R}^2$$

where $n>2$ (so that H has a locally Lipshitz derivative). Then

$$\mathfrak{z} \stackrel{\text{def}}{=} \{z\in\mathbb{R}^2 : H(z)=0\} = \{z\in\mathbb{R}^2 : \|z\|\le 1\}$$



Flattened Hamiltonian (from below)

As usual, define

$$\bar{\nabla} H(x,y) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial H}{\partial y}(x,y) \\ -\frac{\partial H}{\partial x}(x,y) \end{pmatrix}$$

and consider the SDE

$$dZ_t^\varepsilon = \frac{1}{\varepsilon^2}\bar{\nabla} H(Z_t^\varepsilon)dt + dW_t$$

where W is now a *two*-dimensional Brownian motion.

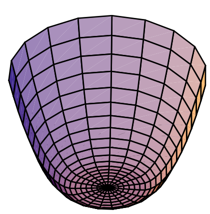
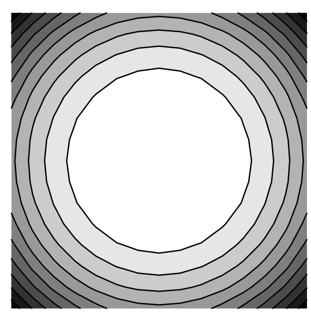
We consider the orbits

$$\begin{aligned}\dot{Z}_t^0(z) &= \bar{\nabla} H(Z_t^0(z)) \\ Z_0^0(z) &= z\end{aligned} \tag{1}$$

and define the measures μ_z by

$$\langle f, \mu_z \rangle \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T} \int_0^T f(Z_s^0(z))ds.$$

As in the classical case, if $H(z) > 0$, then $\mu_z = \mu_{z'}$ if and only if $H(z) = H(z')$. On the other hand, if $H(z) = 0$, then $\mu_z = \mu_{z'}$ if and only if $z = z'$ (each element of $\mathfrak{z}$ is a critical point of (1).

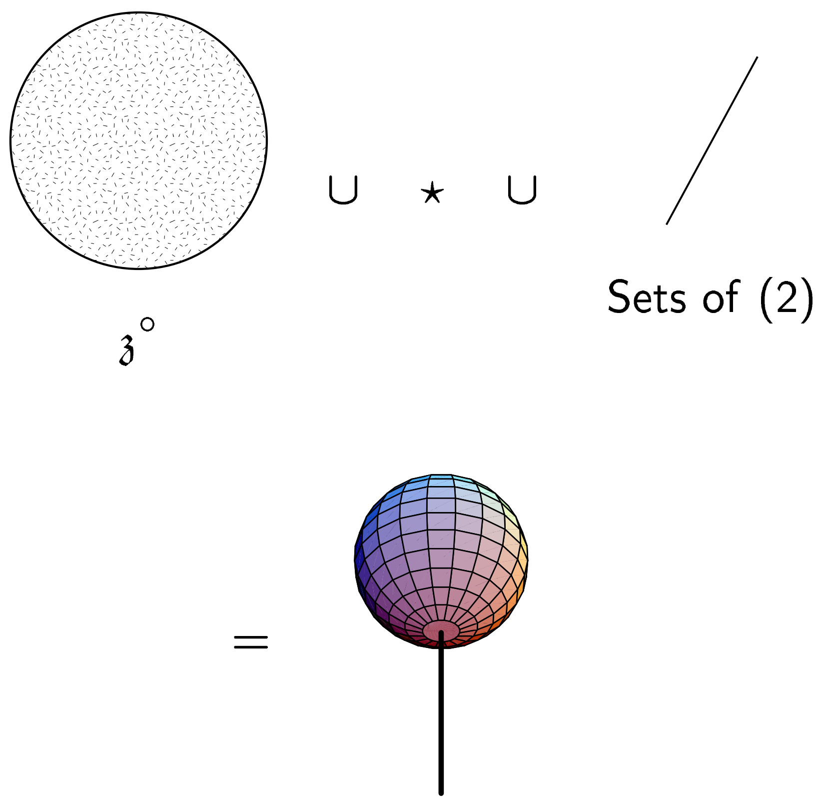
Bottom view of  is 

Contour Plot of Flattened Hamiltonian

Thus, we should collapse each level

$$\{z \in \mathbb{R}^2 : H(z) = h\} \tag{2}$$

to a point if $h > 0$. We should also keep the structure of the (open) disk $\mathfrak{z}^\circ$. We should define a point $\star$ as the edge of the disk and the limit of the sets of (2) as $h \to 0$. This gives us a downward-pointing **lollipop**.

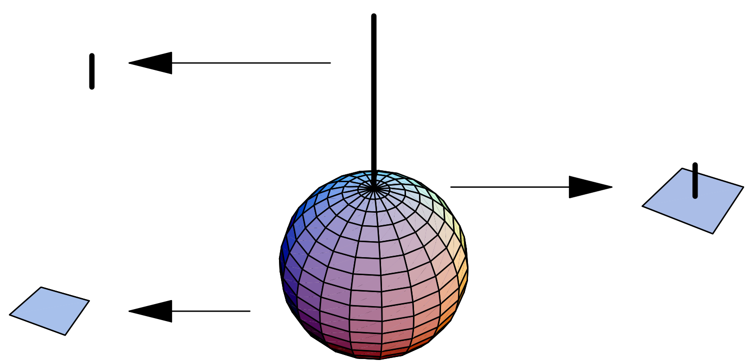


Geometry of Flattened Hamiltonian

Our result is

Theorem. *The process* $[Z_t^\varepsilon]$ *tends to a lollipop-valued Markov process* ξ *with a computable generator.*

The lollipop is a **stratified space**.



Stratified Space

Locally, the lollipop can look like a line, a plane or the union of a line and plane (a flagpole); the lollipop has a **dimensional discontinuity**. This seems to be one of the first instances of a Markov process whose state space is stratified (there is come coeval work by Burdzy and Bass on a “fiber” Brownian motion).

Let's be a bit more precise about our result. The lollipop can be written as

$$L \stackrel{\text{def}}{=} \mathfrak{z}^\circ \cup \star \cup (0,\infty);$$

we map $\{z \in \mathbb{R}^2 : H(z) = h\}$ (for $h > 0$) into $h \in (0,\infty)$. The standard averaging results give us that as long as Z^ε stays outside of $\mathfrak{z}$, the limiting process has generator

$$\mathscr{L}_1 f(h) \stackrel{\text{def}}{=} \frac{1}{2}\sigma^2(h)\ddot{f}(h) + b(h)\dot{f}(h) \qquad f \in C^2(0,\infty)$$

where

$$\sigma^2(h) = \left\langle \|\bar{\nabla} H\|^2, \mu_h \right\rangle \quad \text{and} \quad b(h) = \left\langle \frac{1}{2}\Delta H, \mu_h \right\rangle.$$

When the limiting process is inside $\mathfrak{z}$, it has the same dynamics as the original process (there is no fast drift on $\mathfrak{z}$); the generator inside $\mathfrak{z}$ is thus

$$\mathscr{L}_2 f(h) = \frac{1}{2}\Delta f(h). \qquad f \in C^2(\mathfrak{z}^\circ)$$

We need a **glueing condition** at the junction $\star$. We can motivate (but not prove) the glueing condition via the relevant equation for the density (the forward Kolmogorov equation) of the lollipop-valued process (which we for the moment assume to exist). For explanation, we assume that

$$\mathbb{E}[f(\xi_t)] = \int_{(0,\infty)\subset\mathbb{R}^1} f_1(h)p_1(t,h)dh + \int_{\mathfrak{z}\subset\mathbb{R}^2} f_2(z)p_2(t,z)dz$$

Roughly, the flux through the junction must sum to zero. The flux entering the junction from the handle of the lollipop is one-dimensional. The flux leaving the junction into $\mathfrak{z}$ is the *integral* of the flux at the edge of the disk. Thus we should have

$$-\frac{\partial p_1}{\partial h}(t,0) = \int_{z\in\partial\mathfrak{z}} \frac{\partial p_2}{\partial \nu}(t,z)dz \tag{3}$$

(ν the inward-pointing unit normal at $\partial\mathfrak{z}$). It also seems reasonable that the density is continuous at the junction; i.e.,

$$p_2(t,\cdot)\big|_{\partial\mathfrak{z}} = p_1(t,0). \tag{4}$$

In fact, (the adjoint of) these two conditions (3) and (4) are almost sufficient. The domain of the generator of the lollipop-valued process consists of those functions f such that f_1 and f_2 are C^2 up to the junction and such that

- f is continuous at the vertex
- we have that

$$-\frac{\partial f_1}{\partial h}(0) = \int_{z \in \partial \mathfrak{z}} \frac{\partial f_2}{\partial \nu}(z)dz$$

- $\mathscr{L}_2 f_2(\cdot)\big|_{\partial \mathfrak{z}} = \mathscr{L}_1 f_1(0)$

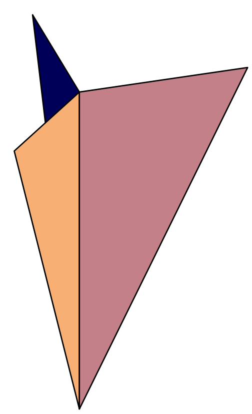
For such a function, we define $\mathscr{L} f$ as $\mathscr{L}_1 f_1$ on $(0,\infty)$ and as $\mathscr{L}_2 f_2$ on $\mathfrak{z}$; by (3), continuity allows us to uniquely define $\mathscr{L} f(\star)$.

The particle diffuses according to different generators on the stick and ball. At the junction, a coin flip decides where the next excursion should be. An *entrance law* is required to start on the ball/disk.

Other Issues

A. General connection between Hamiltonian/classical dynamics and noise.

- Aubry-Mather
- Arnol'd Diffusion
- Higher-dimensional problems



- Resonances

B. Stratified Spaces

- Diffusions
- Natural PDE's (parabolic and elliptic)
- PDE estimates with a separation of scales.